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THE APPLICATION OF FLOQUET THEORY TO THE COMPUTATION OF SMALL ORBITAL PERTURBATIONS OVER LONG TIME INTERVALS USING THE TSCHAUNER-HEMPEL EQUATIONS

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Introduction

This paper deals with a method of calculating the deviation of the path of an orbiting body from a nominal or reference trajectory. The form in which the solution is cast was motivated by a particular perturbation problem. Stanford University is developing a "drag-free", or "drag-makeup", scientific satellite which is designed to follow a purely gravitational orbit.[1] The satellite consists actually of two satellites: an inner sphere or proof mass, and an outer concentric shell. The relative position of the shell with respect to the inner sphere is sensed with a capacitive pickoff. The position signals command an active translation control system which fires jets mounted on the outer shell so that it chases the inner sphere without ever touching it. Thus the proof mass is shielded from gas drag and solar radiation pressure and, except for very small disturbances caused by force interactions with the outer shell, it follows a purely gravitational orbit.

The problem which motivated the present study was to determine the effect of these small disturbances (about 10^{-10} to 10^{-9} g_e) over time periods up to a year. Furthermore, the answer was desired directly in terms of the deviation of the satellite's path from the path which would be followed by an earth satellite acted upon by gravity only. Therefore, the technique of perturbation of the coordinates was selected as the basis of our approach.

The technique of coordinate perturbation, which began with the work of Encke[2] and Hill[3] in the last century, has found increasing use in modern times for orbital theory. The linearized perturbation

equations about a circular orbit (which are merely Hill's lunar equations[3] without the mutual gravitational terms (see equations (1), (2), (3) with $e = 0$) have been applied in recent years by Wheelon,[4] Geyling,[5,6] and Clohessy and Wiltshire[7] to a number of satellite perturbation problems. Battin[8] and Danby[9,10,11] give state transition matrices for general conic sections which also may be applied to satellite perturbation and guidance problems, and recently Tschauner and Hempel[12] have applied the linearized Hill's lunar equations to the minimum fuel rendezvous problem.

In some types of orbital problems (as in the mentioned example of determining the effect of internal force errors on the orbit of a drag-free satellite), it is desirable to compute the perturbations of the coordinates when the satellite is subjected to very small disturbances for many thousands of revolutions. In this case, the linearized Hill's equations are useful only for very very small eccentricities; variation of parameter techniques do not yield an answer directly in the desired form (i.e., as deviations of the coordinates); and direct numerical integration proves both costly and inaccurate, when carried out over long time intervals. Hence a different approach is sought.

The Tschauner and Hempel Equations

Tschauner and Hempel[13] have shown that if the normalized orbit equations of motion are linearized about a nominal elliptical orbit in a rotating reference frame (see Appendix A), they assume the very simple form:

$$\xi'' - \frac{3}{1 + e \cos \theta} \xi - 2\eta' = \alpha \tag{1}$$

$$2\xi' + \eta'' = \beta \tag{2}$$

$$\zeta'' + \zeta = \gamma \tag{3}$$

where $\xi = \frac{u_1}{R}$, $\eta = \frac{u_2}{R}$, $\zeta = \frac{u_3}{R}$,

$\alpha = \frac{P_1}{\omega^2 R}$, $\beta = \frac{P_2}{\omega^2 R}$, $\gamma = \frac{P_3}{\omega^2 R}$

P_1, P_2, P_3 are small perturbing accelerations along the u_1, u_2, u_3 axes respectively,

R is the instantaneous radius of the nominal elliptical orbit ,

θ is the true anomaly in the nominal orbit,

e is the eccentricity of the nominal orbit,

$\omega = \dot{\theta}$, the time rate of change of true anomaly,

u_1, u_2, u_3 are relative coordiantes shown in Figure 1, and

the prime (') signifies $\frac{d}{d\theta} = \frac{1}{\omega}\frac{d}{dt}$.

In deriving these equations, terms of order ξ^2, η^2, ζ^2 and higher are neglected. If the equations of motion in cylindrical form are linearized as shown in Figure 2, with $\xi = \frac{r}{R}$, $\eta = \varphi$, and ζ as before, equations (1) through (3) are again obtained. Now, however, η may be arbitrarily

large while terms of order ξ^2, η^2, ζ^2, and higher are neglected. Equation (3), of course, represents simple out-of-plane harmonic motion and needs no discussion.

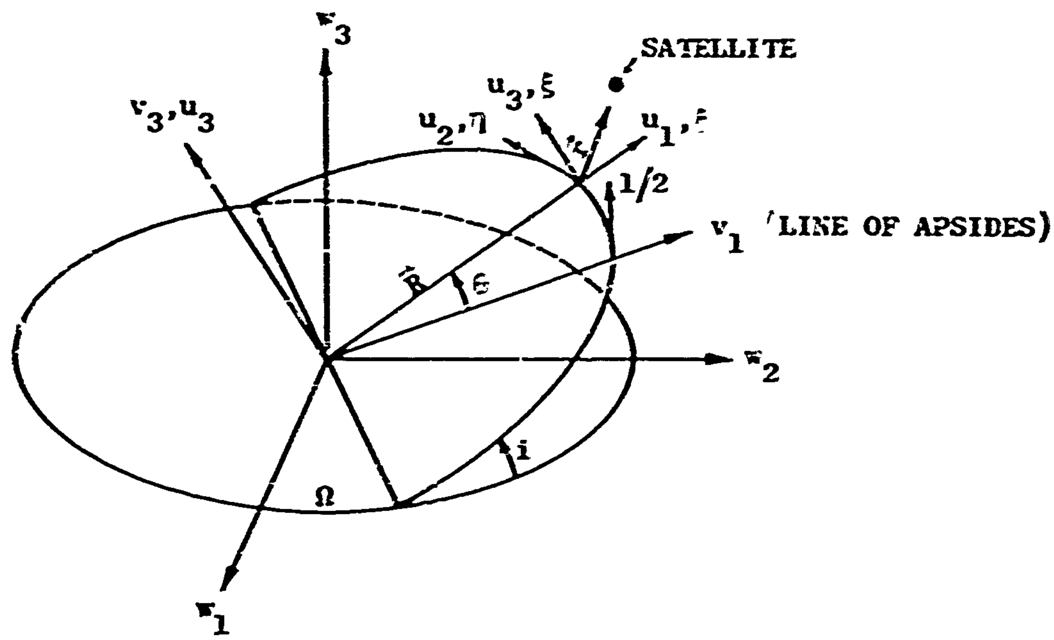

FIG. 1. ORBIT COORDINATE SYSTEM (Rectangular Coordinate Interpretation).

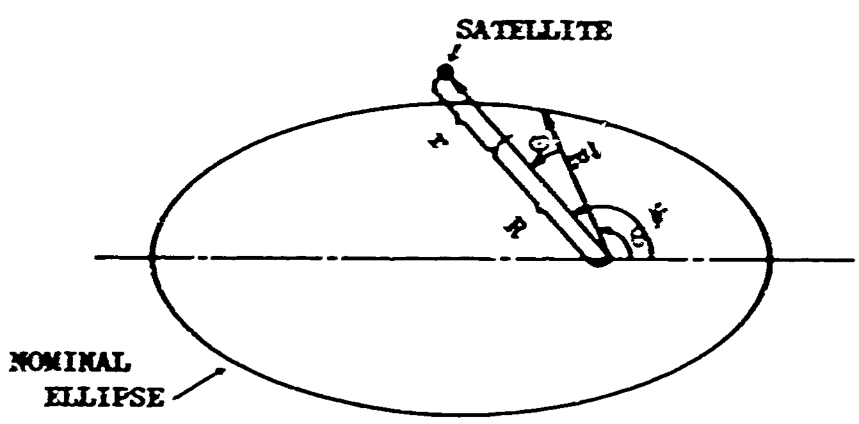

FIG. 2. ORBIT COORDINATE SYSTEM (Cylindr. Coordinate Interpretation).

Solution of the Tschauner-Hempel Equations

By introducing matrix notation and defining the system state matrix $x(\theta)$ to be

$$x(\theta) = \begin{pmatrix} \xi(\theta) \\ \xi'(\theta) \\ \eta(\theta) \\ \eta'(\theta) \end{pmatrix} \tag{4}$$

equations (1) and (2) may be combined and written

$$x'(\theta) = F(\theta)\, x(\theta) + D(\theta)\, u(\theta) \tag{5}$$

where

$$F(\theta) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{3}{1 + e\cos\theta} & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{pmatrix}, \tag{6}$$

$$D(\theta) = \begin{pmatrix} 0 & 0 \\ \frac{1}{\omega^2 R} & 0 \\ 0 & 0 \\ 0 & \frac{1}{\omega^2 R} \end{pmatrix}, \tag{7}$$

and

$$u(\theta) = \begin{pmatrix} P_1(\theta) \\ P_2(\theta) \end{pmatrix} . \tag{8}$$

It is well known from the theory of Floquet[14] (see Appendix B) that a system governed by equation (5) where $F(\theta) = F(\theta + 2\pi)$, has a state transition matrix, $X(\theta, \theta_o)$,* which can be written as:

$$X(\theta, \theta_o) = R(\theta, \theta_o)\, \epsilon^{B(\theta - \theta_o)} \tag{9}$$

where $R(\theta, \theta_o) = R(\theta + 2\pi, \theta_o)$ is a periodic $4 \cdot 4$ matrix, and $B \triangleq \frac{1}{2\pi}\, \ell n\, X(\theta_o + 2\pi, \theta_o)$ is a constant 4×4 matrix whose eigenvalues determine the system stability.

The unforced part of equation (5) is said to be kinematically similar to the constant system

$$w' = Bw . \tag{10}$$

$F(\theta)$, $R(\theta, \theta_o)$ and B are related by

$$B = R^{-1}(\theta, \theta_o)F(\theta)R(\theta, \theta_o) - R^{-1}(\theta, \theta_o)R'(\theta, \theta_o) , \tag{11}$$

and equations (10) and (11) are known as the Lyapunov reduction of equation (5). By an appropriate linear constant transformation

$$z = Qw \tag{12}$$

*Formally, the state transition matrix of an n^{th}-order linear system of differential equations in first-order matrix form is an $n \times n$ matrix whose columns are n linearly independent solutions of the free equation, such that $X'(\theta, \theta_o) = F(\theta)X(\theta, \theta_o)$ and $X(\theta_o, \theta_o) = U$, the unit or identity matrix (see Appendix B).

(where Q is a constant 4 x 4 matrix)

equation (10) may be transformed into its Jordan normal form:

$$z' = \Lambda z \tag{13}$$

where

$$\Lambda = QBQ^{-1} \tag{14}$$

The eigenvalues of Λ, together with the structure of the Jordan blocks determine the stability of the free solution ($u(\theta) = 0$) of equation (5), and it is possible to give the state transition matrix, $X(\theta, \theta_o)$, directly in terms of Λ:

$$X(\theta, \theta_o) = P(\theta)\epsilon^{\Lambda(\theta - \theta_o)}P^{-1}(\theta_o) \tag{15}$$

where $x(\theta) = P(\theta)z(\theta)$. (See Appendix B).

The periodic part of the state transition matrix, $R(\theta, \theta_o)$ is given by

$$R(\theta, \theta_o) = P(\theta)P^{-1}(\theta_o), \tag{16}$$

and furthermore

$$Q = P^{-1}(\theta_o) . \tag{17}$$

It has been shown by Tschauner and Hempel[13] (who have obtained the matrix $P^{-1}(\theta)$ in closed form), and also by the present authors, that equation (10) is kinematically similar to equations (1) and (2) with the Jordan canonical form of B given by

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}^{*} \tag{18}$$

It is rather interesting to note that Λ may be obtained by finding the Jordan canonical form of B_o, where B_o is the matrix $F(\theta)$ given by equation (6) with $e = 0$.**

$$B_o = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{pmatrix} \tag{19}$$

In fact, equation (5) may be factored into the form

$$x'(\theta) = [B_o + e\,G(\theta)]x(\theta) + D(\theta)u(\theta) \tag{20}$$

where $G(\theta) = G(\theta + 2\pi)$,

$$G(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{-3\cos\theta}{1 + e\cos\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{21}$$

---

*The normal form (equation (13) with Λ given by equation (18)) corresponds to two decoupled second-order systems: a pure inertia or $1/s^2$ plant and a harmonic oscillator with a natural period equal to the orbit period. The $1/s^2$ plant may be interpreted physically as motion in a similar coplanar, coaxial ellipse with higher or lower total energy. The harmonic oscillator corresponds to motion in a coplanar ellipse with the same period, but with different eccentricity and/or orientation.

**This remarkable property is not usually possessed by even the simplest of periodic systems. Compare, for example, Mathieu's Equation, $\ddot{\theta} + \omega_o^2(1 - e\cos 2\omega t)\,\theta = 0$.

The matrix $P^{-1}(\theta)$ is given by Tschauner and Hempel:

$$\begin{pmatrix} c_1 & p_2 & \frac{1}{3} & -q_2 \\ -2q_1 + e\mu' & -e\mu & 0 & -q_1 \\ -\frac{1}{2} e \sin\theta & -\frac{1}{2}(1 + e\cos\theta) & \frac{1}{2} e\cos\theta & 0 \\ -\frac{1}{2}(3 + e\cos\theta) & 0 & -\frac{1}{2} e\sin\theta & -\frac{1}{2}(2 + e\cos\theta) \end{pmatrix} \tag{22}$$

where:

$$c_1 = \frac{1}{e}\left[1 - (1 + 2e^2)\sqrt{1-e^2}\right]\sin\theta - (2 + 3e\cos\theta + e^2)\sin^{-1}\lambda, \tag{23}$$

$$p_2 = \frac{-1}{6}(1 + 3\sqrt{1-e^2}) - \frac{1}{3e}\left[1 - (1 - e^2)^{3/2}\right]\cos\theta$$

$$+ \frac{1}{6}\left[(1 + 2e^2)\sqrt{1-e^2} - 1\right]\cos 2\theta - e\mu\sin^{-1}\lambda, \tag{24}$$

$$q_2 = \frac{1}{3e}\left[(2 + e^2)\sqrt{1-e^2} - 2\right]\sin\theta + \frac{1}{6}\left[(1 + 2e^2)\sqrt{1-e^2} - 1\right]\sin 2\theta$$

$$+ (1 + e\cos\theta)^2\sin^{-1}\lambda, \tag{25}$$

$$q_1 = (1 + e\cos\theta)^2, \tag{26}$$

$$\mu = \sin\theta(1 + e\cos\theta), \tag{27}$$

$$\lambda = \frac{\sin\theta\left[e + (1 - \sqrt{1-e^2})\cos\theta\right]}{1 + e\cos\theta} \tag{28}$$

If θ is chosen to be zero, then

$$
P^{-1}(0) = \begin{pmatrix} 0 & \dfrac{(e+1)\left[-1+(1-e)^2\sqrt{1-e^2}\right]}{3e} & \dfrac{1}{3} & 0 \\[2ex] -(1+e)(2+e) & 0 & 0 & -(1+\ )^2 \\[2ex] 0 & -\frac{1}{2}(1+e) & \frac{1}{2}e & 0 \\[2ex] -\frac{1}{2}(3+e) & 0 & 0 & -\frac{1}{2}(2+e) \end{pmatrix} \tag{29}
$$

$$
P(0) = \begin{pmatrix} 0 & -\dfrac{2+e}{1+e} & 0 & 2(1+e) \\[2ex] \dfrac{3e}{(1-e)(1-e^2)^{3/2}} & 0 & \dfrac{-2}{(1-e)(1-e^2)^{3/2}} & 0 \\[2ex] \dfrac{3}{(1-e)^2\sqrt{1-e^2}} & 0 & \dfrac{-2\left[1-(1-e)^2\sqrt{1-e^2}\right]}{e(1-e)^2\sqrt{1-e^2}} & 0 \\[2ex] 0 & \dfrac{3+e}{1+e} & 0 & -2(2+e) \end{pmatrix} \tag{30}
$$

From equations (14) and (17)

$$
B(e) = P(0)A\,P^{-1}(0) \tag{31}
$$

so that

and also $B_o = B(0)$.

It may be seen by direct differentiation that the solution to equation (5) is

$$x(\theta) = X(\theta, \theta_o)x(\theta_o) + X(\theta, \theta_o) \int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \qquad (33)$$

where $x(\theta_o)$ is the initial value of the system state matrix and

$$X(\theta, \theta_o) = R(\theta, \theta_o)\epsilon^{B(\theta-\theta_o)} = P(\theta)\epsilon^{\Lambda(\theta-\theta_o)}P^{-1}(\theta_o) \ . \qquad (34)$$

(This solution may also be obtained by variation of parameters. See Appendix B) If one attempts to use equation (33) directly to determine the effect of small perturbing accelerations over many revolutions, serious numerical difficulties are encountered which result both in loss of accuracy and in excessive computation time.

$$
B(e) = \begin{pmatrix}
0 & (1+e)^2 & -e(1+e) & 0 \\[2ex]
\dfrac{3-5e-9e^2-e^3}{(1-e)(1-e^2)^{3/2}} & 0 & 0 & \dfrac{2-2e-6e^2-3e^3}{(1-e)(1-e^2)^{3/2}} \\[3ex]
\dfrac{(3-5e-9e^2-e^3)-(3-5e+e^2+e^3)\sqrt{1-e^2}}{e(1-e)^2\,(1-e^2)} & 0 & 0 & \dfrac{(2-2e-6e^2-3e^3)-(2-3e+e^3)\sqrt{1-e^2}}{e(1-e)^2\sqrt{1-e^2}} \\[3ex]
0 & -(1+e)(2+e) & -e(2+e) & 0
\end{pmatrix} \qquad (32)
$$

Solution for Constant, Periodic, and Almost-Periodic Perturbing Accelerations

As in the case of computing perturbations for a drag-free satellite, it often happens that perturbing accelerations are constant or periodic. It is then possible to compute their effect at any future time merely by computing their effect over one orbit revolution. If the disturbing acceleration has the form

$$u(\theta) = u(\theta + 2\pi) \tag{35}$$

it can be shown (see Appendix C) that the solution to equation (5) [that is, equation (33)] can be written

$$\begin{aligned} x(\theta) = {} & X(\theta - 2\pi N, \theta_o) C^N x(\theta_o) + X(\theta - 2\pi N, \theta_o)\left(\sum_{k=1}^{N} C^k\right) I_1 \\ & + X(\theta - 2\pi N, \theta_o) \int_{\theta_o}^{\theta - 2\pi N} X^{-1}(\tau, \theta_o) D(\tau) u(\tau) d\tau \end{aligned} \tag{36}$$

where N is the largest number of complete revolutions in $(\theta - \theta_o)$, $C \triangleq X(\theta_o + 2\pi, \theta_o)$, and

$$I_1 = \int_{\theta_o}^{\theta_o + 2\pi} X^{-1}(\tau, \theta_o) D(\tau) u(\tau) d\tau. \tag{37}$$

The solution as given by equations (36) and (37) requires integration over a maximum of one orbit revolution, regardless of the actual number of orbit revolutions contained in the range of interest $(\theta - \theta_o)$. Thus the difficulties mentioned in the application of the solution in the form given by equation (33) are overcome. The restriction of the disturbance

to constant or periodic in θ case can be relaxed somewhat. If

$$u(\theta) = u(\theta + 2\pi M) \tag{38}$$

where M is an integer, it can be shown (see Appendix C) that the solution [equation (33)] can be written

$$x(\theta) = X(\theta - 2\pi N, \theta_o)C^N x(\theta_o) + X(\theta - 2\pi N, \theta_o)\left(\sum_{k=0}^{r-1} C^{N-kM}\right) I_2$$

$$+ X(\theta - 2\pi N, \theta_o)C^{N-rM} \int_{\theta_o}^{\theta - 2\pi rM} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \tag{39}$$

where N is the largest number of complete revolutions in $(\theta - \theta_o)$,

r is the largest integer $\leq N/M$,

$C = X(\theta_o + 2\pi, \theta_o)$, and

$$I_2 = \int_{\theta_o}^{\theta_o + 2\pi M} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \tag{40}$$

In this case the solution over any interval $(\theta - \theta_o)$ requires integration over a maximum of M revolutions. Thus the constant r defined above is a figure-of-merit for the solution in this form. The larger r, the more relative value equation (39) has over equation (33).

One further generalization of the form of the perturbing acceleration can be made. If instead of equations (35) or (38) we have

$$u(\theta) = u(\theta + \Theta) \tag{41}$$

where $\Theta \neq 2\pi M$ for $M = 0, 1, 2, \ldots$, the solution may be approximated (again see Appendix C) as closely as desired by selecting an integer K

such that

$$K \epsilon \cong 2\pi M \tag{42}$$

for some integer M. Then the solution to equation (5) is again equation (39), with N, C, and I_2 as defined, but with r an integer such that

$$r K \epsilon \leq 2\pi N < (r + 1) K \epsilon \tag{43}$$

In this final case integration is required over a maximum of M revolutions. Of course, the larger the selected value of M, the greater the accuracy obtained in the approximation of equation (42). The usefulness of the solution, in this case, is dependent upon the nature of the actual problem.

Restriction of Initial True Anomaly to Zero

If the initial value of the true anomaly is taken to be zero ($\theta_o = 0$), no real restriction of the general problem is imposed. This is so because stipulation of $\theta_o = 0$ simply requires a compensatory adjustment in the initial value of the system state matrix $x(\theta_o)$. Then the solution for perturbing accelerations of the form

$$u(\theta) = u(\theta + 2\pi) \tag{35}$$

can be written in a manner especially adapted for rapid, accurate evaluation. If $\theta_o = 0$ and $x(\theta_o) \triangleq x_o$, equations (36) and (37) become

$$x(\theta) = X(\sigma,0)X^{N}(2\pi,0)x_o + X(\sigma,0)\left(\sum_{k=1}^{N} X^{k}(2\pi,0)\right)I$$

$$+ X(\sigma,0)\int_0^{\sigma} X^{-1}(\tau,0)D(\tau)u(\tau)d\tau \tag{44}$$

where N is the largest number of complete revolutions in θ,

$$\sigma = \theta - 2\pi N \quad , \tag{45}$$

$$I = \int_0^{2\pi} X^{-1}(\tau,0)D(\tau)u(\tau)d\tau \tag{46}$$

$$X(2\pi,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-6\pi e(2+e)}{(1-e)^2\sqrt{1-e^2}} & 1 & 0 & \frac{-6\pi e(1+e)}{(1-e)^2\sqrt{1-e^2}} \\ \frac{-6\pi(2+e)(1+e)}{(1-e)^2\sqrt{1-e^2}} & 0 & 1 & \frac{-6\pi(1+e)^2}{(1-e)^2\sqrt{1-e^2}} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{47}$$

$$X^{-1}(\sigma,0) = \begin{pmatrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{pmatrix} , \tag{48}$$

where

$$(1,1) = \frac{4 + e - 3\cos\sigma}{1 + e} , \tag{49}$$

$$(1,2) = \frac{-\sin\sigma\,(1 + e\cos\sigma)}{1 + e} , \tag{50}$$

$$(1,3) = (2,3) = (4,3) = 0 , \tag{51}$$

$$(1,4) = \frac{2 + e - \cos\sigma(2 + e\cos\sigma)}{1 + e} , \tag{52}$$

$$(2,1) = \frac{3e(2+3e\cos\sigma + e^2)(\sigma-\sin^{-1}\lambda) - (3+6e^2)\sqrt{1-e^2}\,\sin\sigma}{(1 - e)(1 - e^2)^{3/2}} , \tag{53}$$

$$(2,2) = \frac{3e^2\sin\sigma(1+e\cos\sigma)(\sigma-\sin^{-1}\lambda)-\sqrt{1-e^2}\,[2e+e^3-(1-e^2)\cos\sigma-(e+2e^3)\cos^2\sigma]}{(1 - e)(1 - e^2)^{3/2}} \tag{54}$$

$$(2,4) = \frac{3e(1+e\cos\sigma)^2(\sigma-\sin^{-1}\lambda)-\sqrt{1-e^2}\,[(2+e^2)\sin\sigma+(e+2e^3)\sin\sigma\cos\sigma]}{(1 - e)(1 - e^2)^{3/2}} , \tag{55}$$

$$(3,1) = \frac{3(2+3e\cos\sigma+e^2)(\sigma-\sin^{-1}\lambda)-(6+3e)\sqrt{1-e^2}\,\sin\sigma}{(1 - e)^2\sqrt{1 - e^2}} , \tag{56}$$

$$(3,2) = \frac{3e\sin\sigma(1+e\cos\sigma)(\sigma-\sin^{-1}\lambda)-\sqrt{1-e^2}\,[2+e^2-(2-2e)\cos\sigma-(2e+e^2)\cos^2\sigma]}{(1 - e)^2\sqrt{1 - e^2}} , \tag{57}$$

$$(3,3) = 1 , \tag{58}$$

$$(3,4) = \frac{3(1+e\cos\sigma)^2(\sigma-\sin^{-1}\lambda)-\sqrt{1-e^2}\,[(4-e)\sin\sigma+(2e+e^2)\sin\sigma\cos\sigma}{(1-e)^2\sqrt{1-e^2}} \quad . \tag{59}$$

$$(4,1) = \frac{6(\cos\sigma - 1)}{1+e} \quad , \tag{60}$$

$$(4,2) = \frac{2\sin\sigma(1+e\cos\sigma)}{1+e} \quad , \tag{61}$$

$$(4,4) = \frac{-(3+e)+2\cos\sigma(2+e\cos\sigma)}{1+e} \quad , \tag{62}$$

and

$$\lambda = \frac{\sin\sigma[e+(1-\sqrt{1-e^2})\cos\sigma]}{1+e\cos\sigma} \quad . \tag{63}$$

If J is the Jordan canonical form of $X(2\pi,0)$ (given by equation (47)), then J is given by

$$J = \begin{pmatrix} 1 & 2\pi & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{64}$$

and (see Appendix B)

$$X(2\pi,0) = P(0)JP^{-1}(0) \tag{65}$$

where $P(0)$ and $P^{-1}(0)$ are given by equations (29) and (30). Noting then that

$$X^N(2\pi,0) = P(0)J^NP^{-1}(0) \quad , \tag{66}$$

and defining

$$S = \sum_{k=1}^{N} J^k \quad , \tag{67}$$

equation (44) can be written in a form which is convenient for calculating $x(\theta)$ when N is large:

$$x(\theta) = X(\sigma,0)P(0)J^N P^{-1}(0) + X(\sigma,0)P(0)SP^{-1}(0)I$$

$$+ X(\sigma,0)\int_0^{\sigma} X^{-1}(\tau,0)D(\tau)u(\tau)d\tau \tag{70}$$

where N is the largest number of complete revolutions in θ,

$X^{-1}(\tau,0)$ is obtained from equations (48) through (63),

$D(\tau)$ is given by equation (7),

$u(\tau)$ is given by equation (35),

$$J^N = \begin{pmatrix} 1 & 2\pi N & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \tag{71}$$

$$S = \begin{pmatrix} N & N(N+1)\pi & 0 & 0 \\ 0 & N & 0 & 0 \\ 0 & 0 & N & 0 \\ 0 & 0 & 0 & N \end{pmatrix} \quad . \tag{72}$$

Sampled-Data Solution

If in the general solution (equation (70)) τ is restricted to zero, an expression is obtained which represents sampled values of the perturbed motion taken at intervals of 2π:

$$x(2\pi N) = P(0)J^{N} P^{-1}(0)x_{o} + P(0)\ S\ P^{-1}(0)I \tag{73}$$

where, from equations (29), (30), (71) and (72) we obtain

$$P(0)J^{N}P^{-1}(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\dfrac{6e\pi(2+e)N}{(1-e)^{2}\sqrt{1-e^{2}}} & 1 & 0 & -\dfrac{6e\pi(1+e)N}{(1-e)^{2}\sqrt{1-e^{2}}} \\ -\dfrac{6\pi(1+e)(2+e)N}{(1-e)^{2}\sqrt{1-e^{2}}} & 0 & 0 & -\dfrac{6\pi(1+e)^{2}N}{(1-e)^{2}\sqrt{1-e^{2}}} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{74}$$

and

$$P(0)SP^{-1}(0) = \begin{pmatrix} N & 0 & 0 & N \\ -\dfrac{3e\pi(2+e)(N+1)N}{(1-e)^{2}\sqrt{1-e^{2}}} & N & 0 & -\dfrac{3e\pi(1+e)(N+1)N}{(1-e)^{2}\sqrt{1-e^{2}}} \\ -\dfrac{3\pi(1+e)(2+e)(N+1)N}{(1-e)^{2}\sqrt{1-e^{2}}} & 0 & N & -\dfrac{3\pi(1+e)^{2}(N+1)N}{(1-e)^{2}\sqrt{1-e^{2}}} \\ 0 & 0 & 0 & N \end{pmatrix}, \tag{75}$$

and where I is defined by equation (46),

$x_o = x(0)$ the initial value of the system state matrix.

The I-matrix has been evaluated (primarily by contour integration) for the case of accelerations constant in the rotating reference frame; that is, for accelerations of the form

$$u(\theta) = \begin{pmatrix} a_\xi \\ a_\eta \end{pmatrix} \quad (a_\xi,\ a_\eta \text{ constant}) \tag{76}$$

The result of this evaluation is contained in Appendix D. Using this result one obtains for accelerations described by equation (76):

$$P(0)SP^{-1}(0)I = \frac{1}{k/p^2} \begin{pmatrix} \frac{(4-e)\pi N}{(1-e)(1-e^2)^{3/2}} a_\eta & \\ -\frac{3e(e^2+2e+2)\pi N}{(1-e)(1-e^2)^{5/2}} a_\xi & -\frac{6e\pi^2 N^2}{(1-e)(1-e^2)^2} a_\eta \\ -\frac{(e^2+10e+4)\pi N}{(1-e)^2(1-e^2)^{3/2}} a_\xi & -\frac{6\pi^2 N^2}{(1-e)^2(1-e^2)} a_\eta \\ \frac{3(e^2-2e-2)\pi N}{(1-e^2)^{5/2}} a_\eta & \end{pmatrix} \tag{77}$$

where k is the gravitational field constant

p is the semilatisrectum of the nominal ellipse

A closed-form of the I-matrix has also been obtained for

accelerations of the form

$$u(\theta) = \begin{pmatrix} c_1 \varepsilon^{jK_1\theta} \\ \\ c_2 \varepsilon^{jK_2\theta} \end{pmatrix} \tag{78}$$

where K_1 and K_2 are integers,

c_1 and c_2 are arbitrary complex constants.

This result is also presented in Appendix D. It is possible then, using the two forms of the I-matrix (for constant and for periodic accelerations), to derive an appropriate I-matrix for any disturbance which can be expanded in a Fourier series in true anomaly.

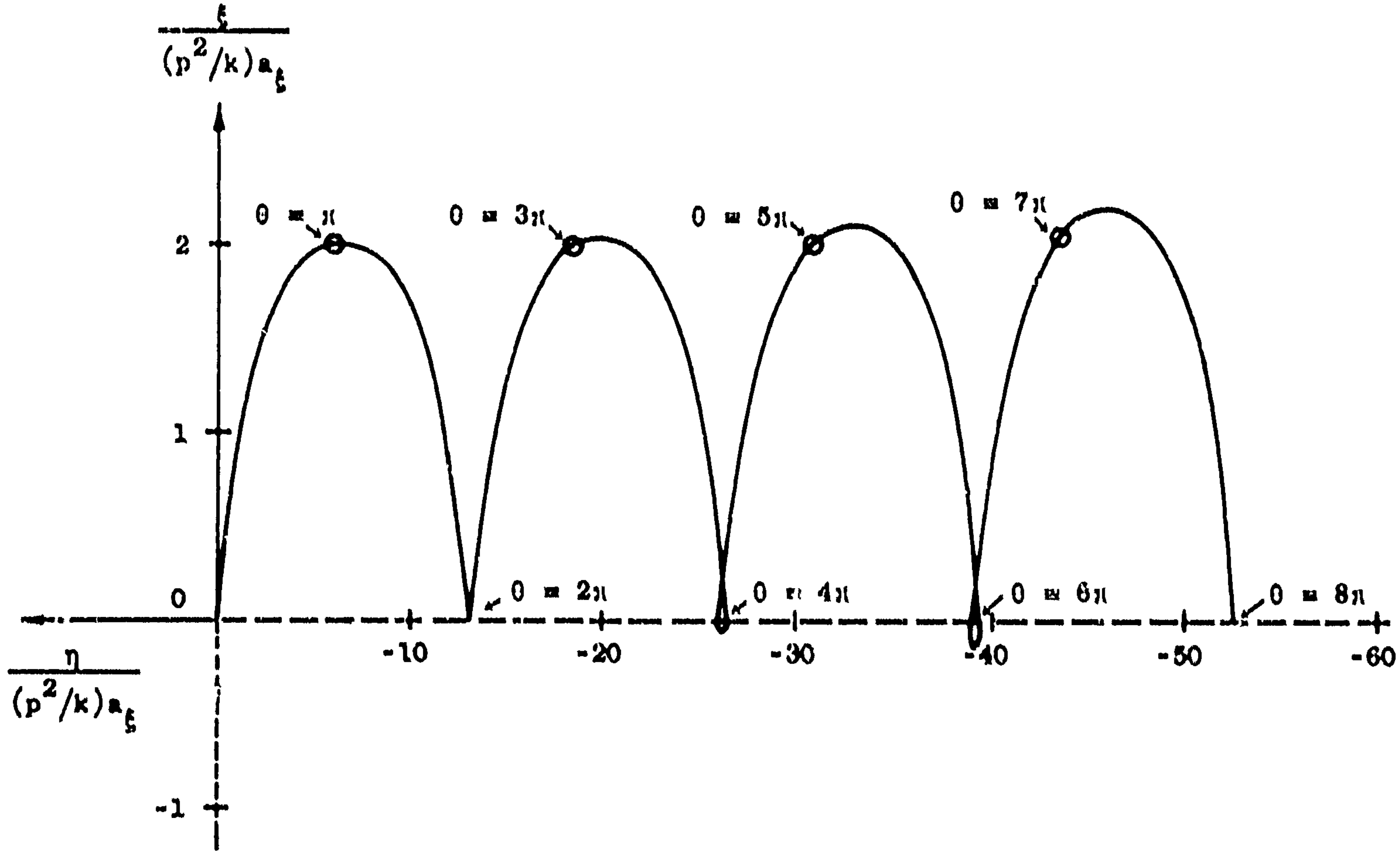

FIG. 3. ACCELERATION CONSTANT IN ROTATING REFERENCE FRAME

$a_\xi \neq 0$; $a_\eta = 0$; $o = 0.01$

Application to the Drag-Free Satellite

As mentioned in the Introduction, the proof-mass or inner sphere portion of the drag-free satellite follows a pure gravity orbit except for very small perturbations caused by force interactions between the inner and outer satellites. The majority of these force interactions are essentially fixed within the satellite. When the satellite is maintained in a locally-level orientation these forces are then fixed in the rotating reference frame and can be described by equation (76), i.e.,

$$a(\theta) = \begin{pmatrix} a_\xi \\ \\ a_\eta \end{pmatrix} \quad (a_\xi, a_\eta \text{ constant}) \tag{76}$$

Plots of typical perturbed motions, assuming zero initial conditions, resulting from accelerations of this type (for selected values of nominal orbit eccentricity) are presented in Figures (3) through (6). It is interesting to compare these plots with Figures (4-5) through (4-8) of Reference 15, which represent the solutions for zero eccentricity. As would be expected, the results for $e = .01$ are almost identical to those for $e = 0$, but do exhibit the trend or distortion shown amplified in the plots for $e = .1$. The effect of the secular terms of the solution are most easily obtained through the sampled-data solution of equation (73). Again ignoring initial condition effects, and selecting for example $e = .01$, then, using equation (77), one has

$$\xi(2\pi N) \cong \frac{4\pi N}{k/p^2} a_\eta \tag{79}$$

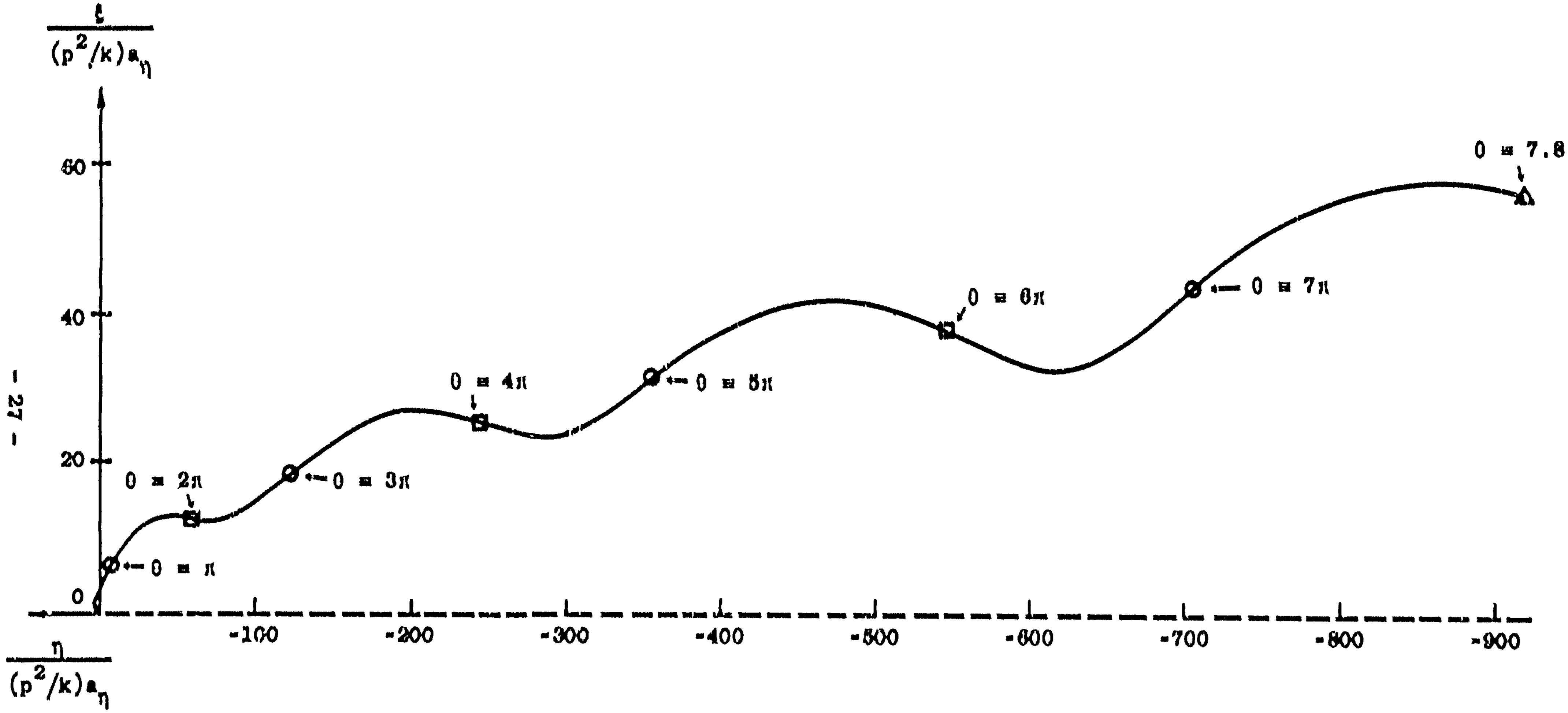

FIG. 5. ACCELERATION CONSTANT IN ROTATING REFERENCE FRAME

$a_\xi = 0$; $a_\eta \neq 0$; $e = 0.01$

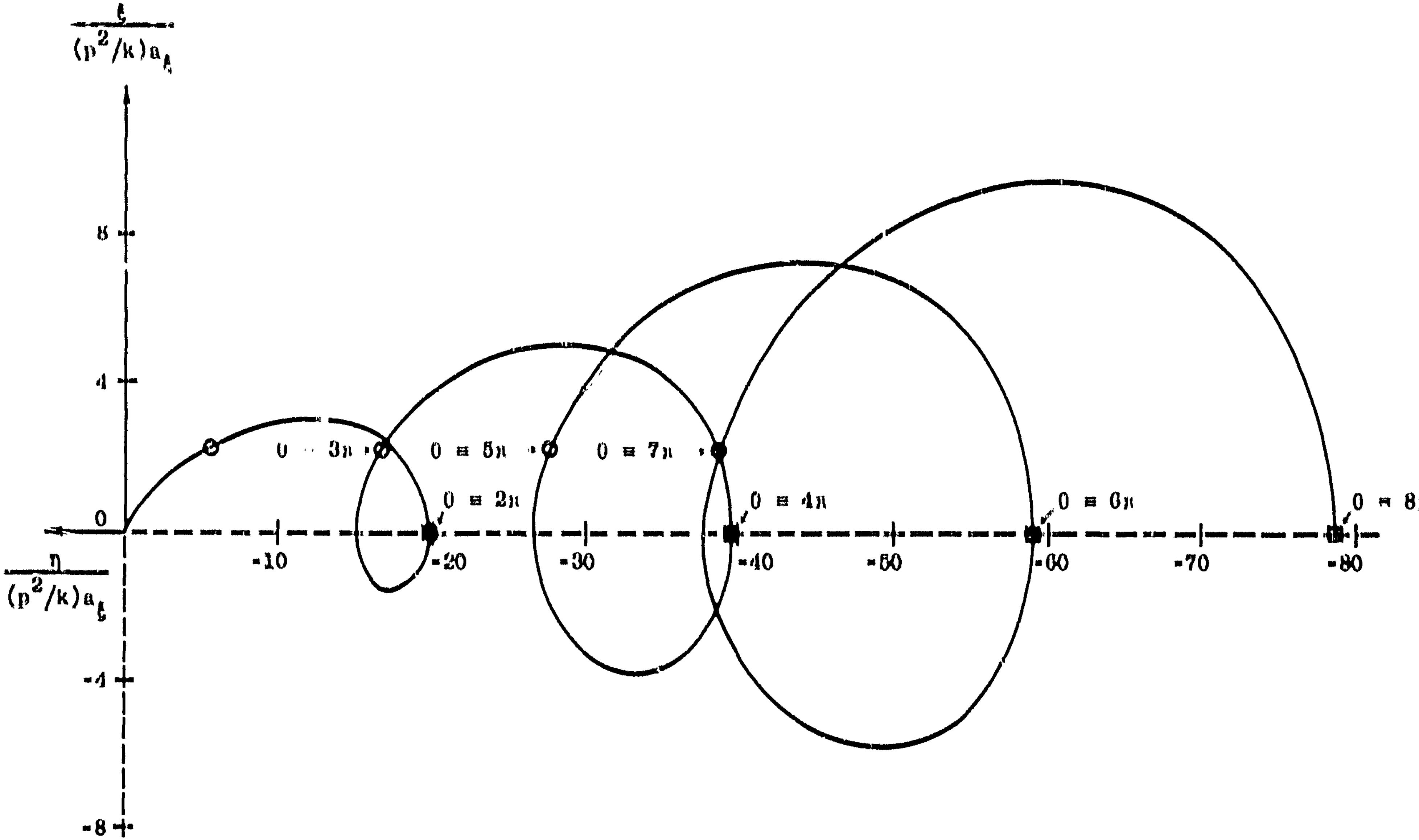

FIG. 4. ACCELERATION CONSTANT IN ROTATING REFERENCE FRAME

$a_\xi \neq 0$; $a_\eta = 0$; $e = 0.1$

$$\eta(2\pi N) \cong -\frac{4\pi N}{k/p^2} a_\xi - \frac{6\pi^2 N^2}{k/p^2} a_\eta \tag{80}$$

If $a_\xi = a_\eta \approx 10^{-10}$ m/sec^2, $p \approx 100$ miles, plus the radius of the Earth, $N \approx 6000$rev (≈ 1 year), and using the basic relationships $x = \xi R$, $y = \eta R$, it is seen that

$$x(1\ \text{year}) \cong 6\ \text{m} \cong 20\ \text{feet} \tag{81}$$

$$y(1\ \text{year}) \cong -10^5\ \text{m} \cong -60\ \text{miles} \tag{82}$$

These results verify those of page (126) of Reference 15.

The drag-free satellite may also be oriented so that it maintains its orientation with respect to inertial space. Then the perturbing acceleration would be essentially fixed in inertial space. If it is resolved into a component $a_{\xi o}$ lying along the line of apsides of the nominal orbit and positive outward (away from the focus) and a component $a_{\eta o}$ perpendicular to $a_{\xi o}$ in the plane of the nominal orbit,* and positive in the direction of motion, then the acceleration vector becomes

$$u(\theta) = \begin{bmatrix} a_{\xi o} \cos\theta + a_{\eta o} \sin\theta \\ \\ -a_{\xi o} \sin\theta + a_{\eta o} \cos\theta \end{bmatrix} \tag{83}$$

where $a_{\xi o}$ and $a_{\eta o}$ are constant. Examples of typical motion are presented in Figures (7) through (10). Again it is interesting to compare these results with those for zero eccentricity contained in Reference 15

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*As was noted previously, out of plane motion is simple, decoupled, harmonic motion, requiring no discussion.

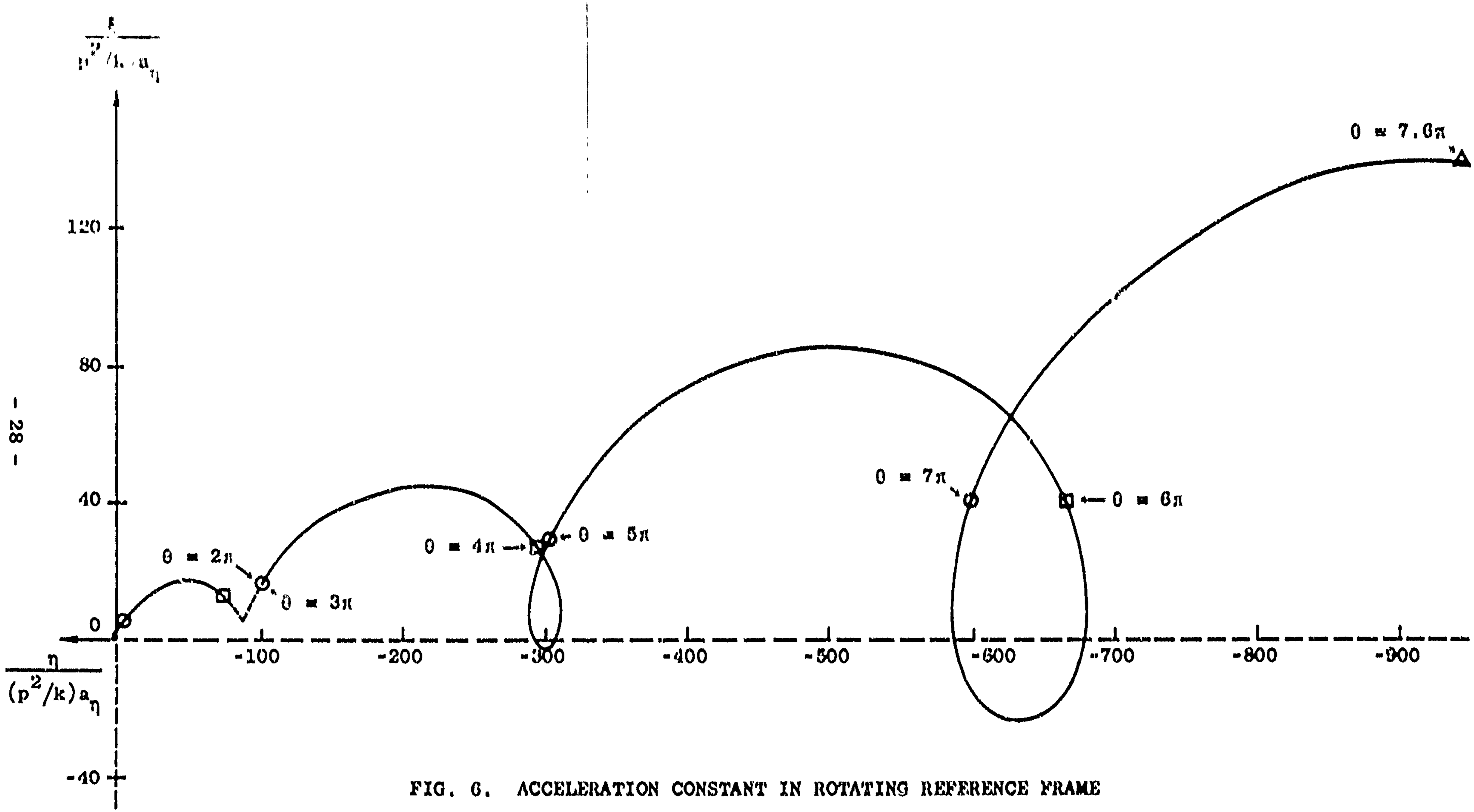


FIG. 6. ACCELERATION CONSTANT IN ROTATING REFERENCE FRAME

$a_\xi = 0$; $a_\eta \neq$; $e = 0.1$

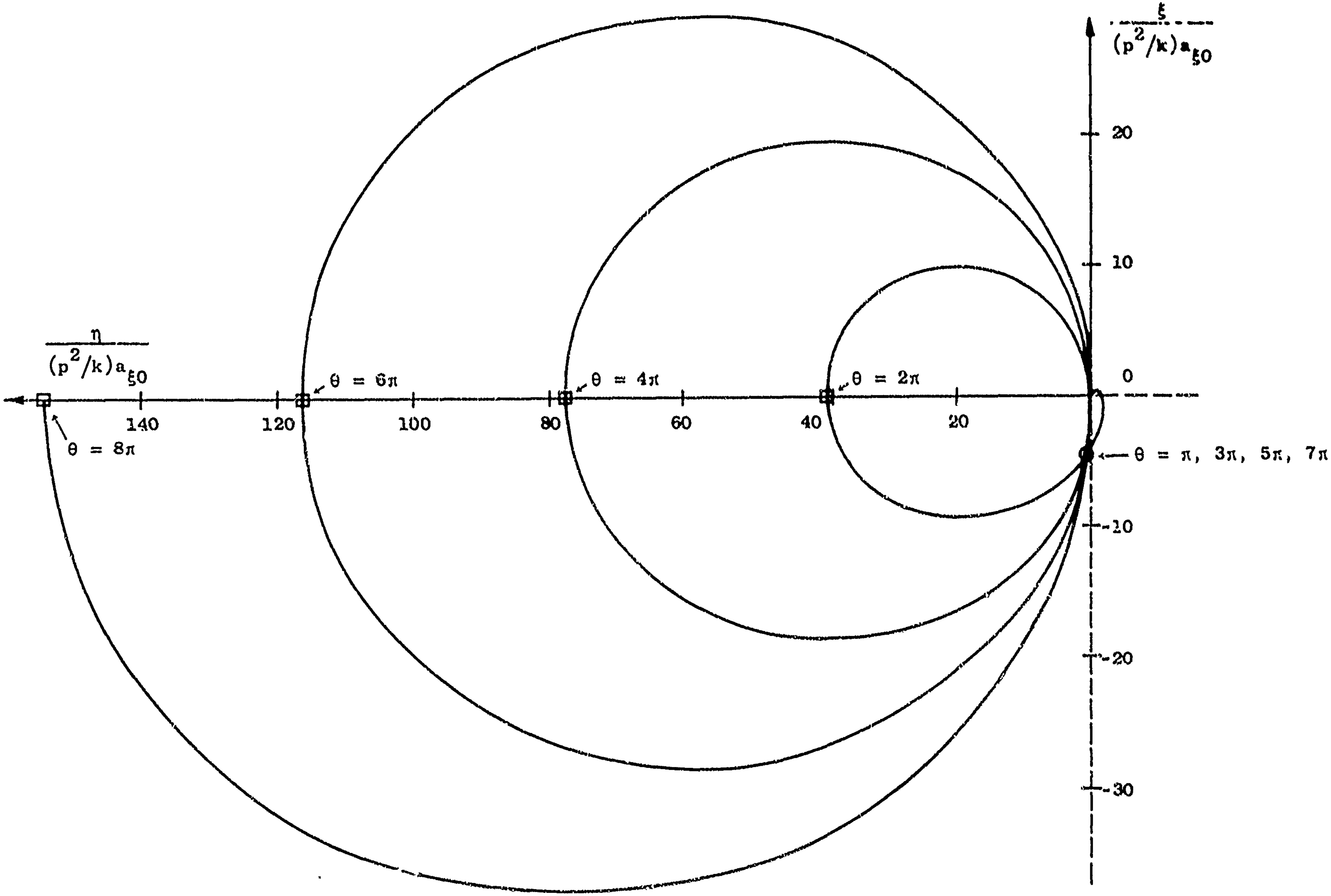


FIG. 7. ACCELERATION CONSTANT IN INERTIAL SPACE

$a_{\xi o} \neq 0$; $a_{\eta o} = 0$; $e = 0.01$

as Figures (4-9) and (4-10).

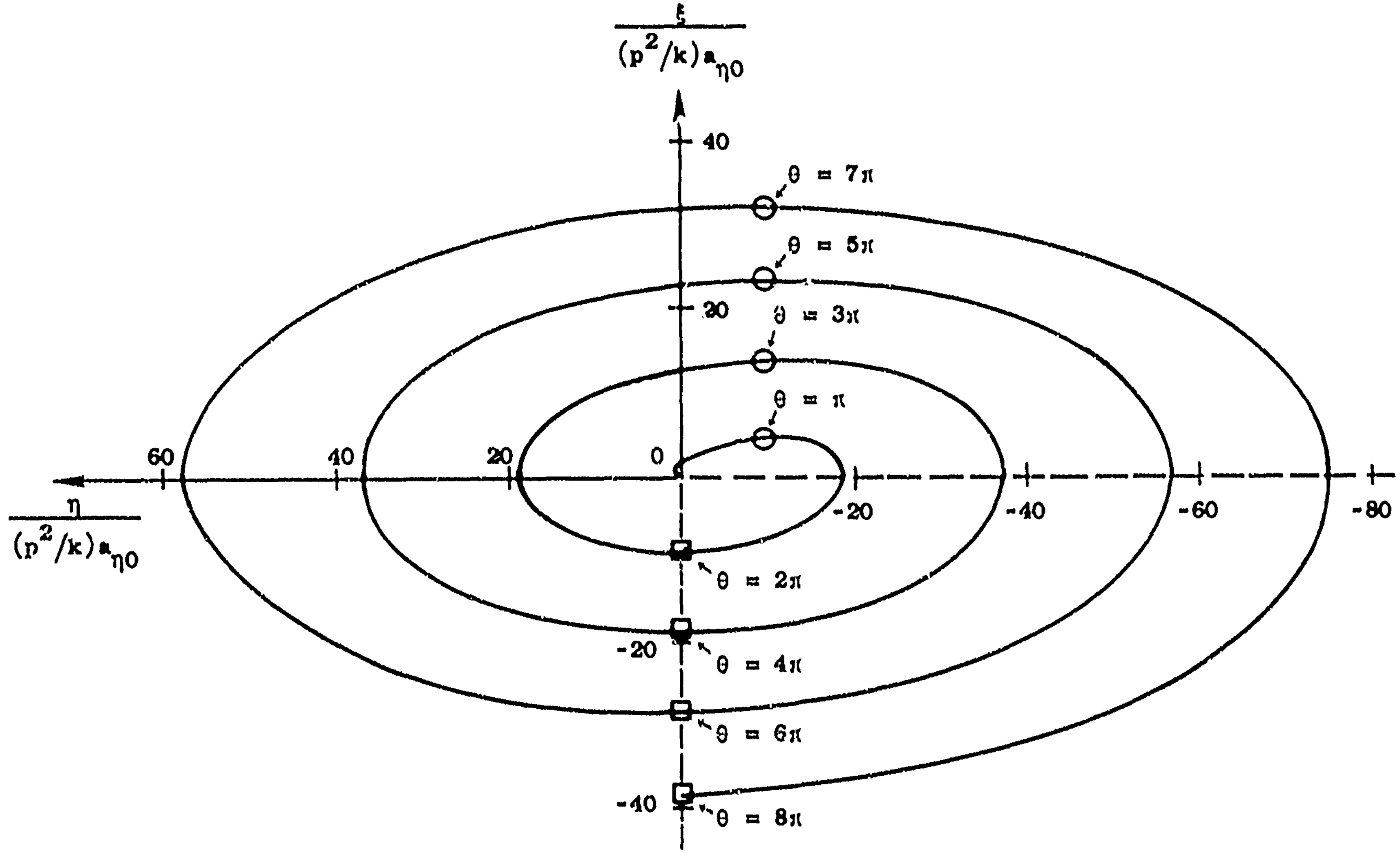


FIG. 9. ACCELERATION CONSTANT IN INERTIAL SPACE

$a_{\xi o} = 0$; $a_{\eta o} \neq 0$; $e = 0.01$

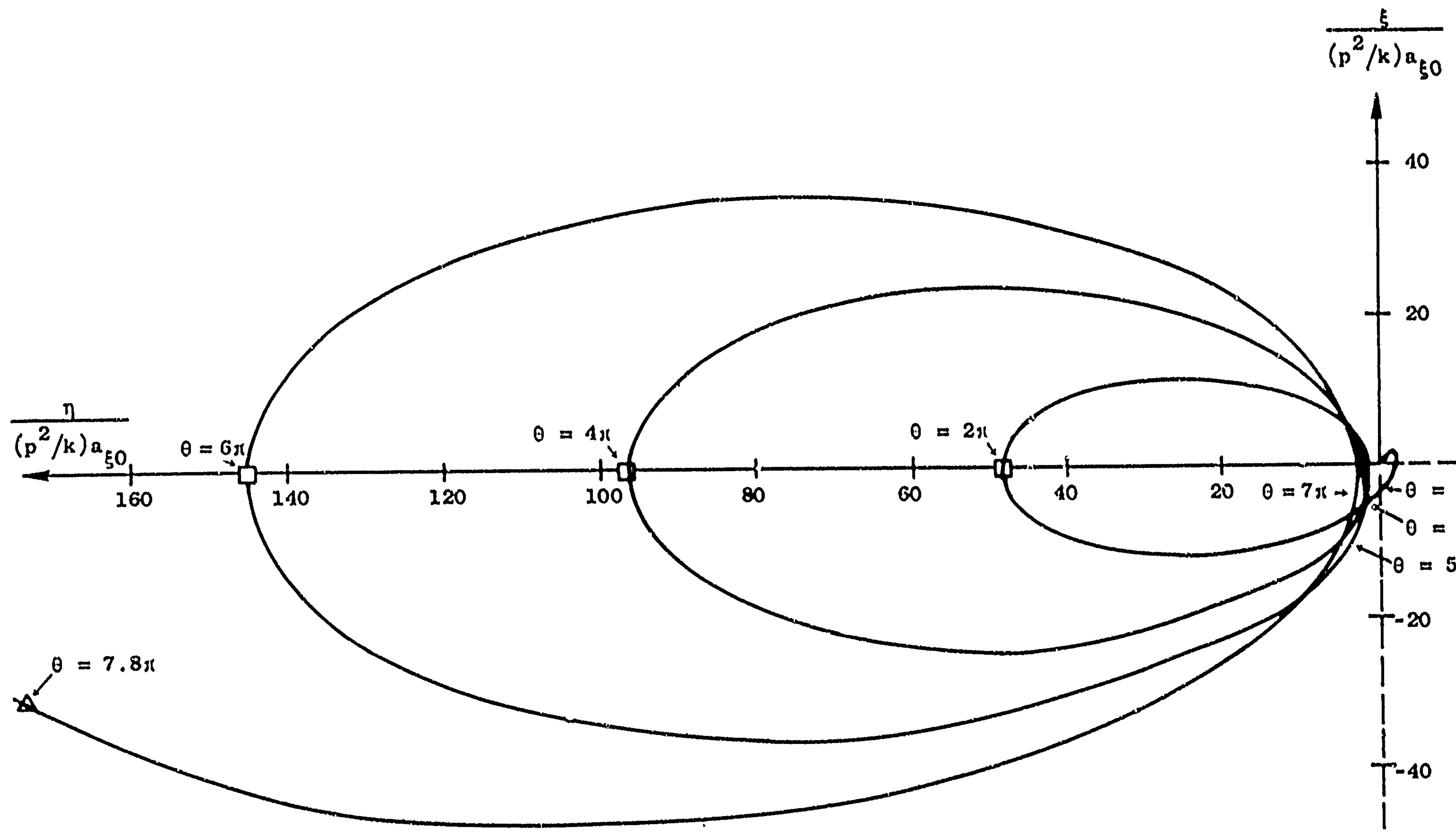


FIG. 8. ACCELERATION CONSTANT IN INERTIAL SPACE

$a_{\xi o} \neq 0$; $a_{\eta o} = 0$; $e = 0.1$

Application to the Problem of Solar Radiation With Shadowing

As an example of how periodic disturbances might arise with an ordinary satellite, consider an approximate solution of the solar radiation pressure problem where the sun is assumed to remain fixed with respect to the orbit plane. If this perturbation is desired over a relatively few orbit periods, then it is reasonable to regard the disturbing acceleration as essentially fixed in inertial space. The reference orbit would, of course, be perturbed by the earth's oblateness, but over a few orbit periods this will not result in very great relative motion of the sun.

If θ_i is the true anomaly when the satellite enters the shadow, and θ_o the corresponding exit value (see Figure (11)), then

$$u(\theta) = \begin{bmatrix} a_{\xi o} \cos\theta + a_{\eta o} \sin\theta \\ \\ -a_{\xi o} \sin\theta + a_{\eta o} \cos\theta \end{bmatrix}, \quad \begin{matrix} (0 \leq \theta < \theta_i) \\ \\ (\theta_o < \theta \leq 2\pi) \end{matrix} \tag{84}$$

$$u(\theta) = 0 \quad , \quad \theta_i \leq \theta \leq \theta_o \tag{85}$$

where the acceleration vector $\vec{a}$ has been resolved as was done previously for accelerations fixed in inertial space.

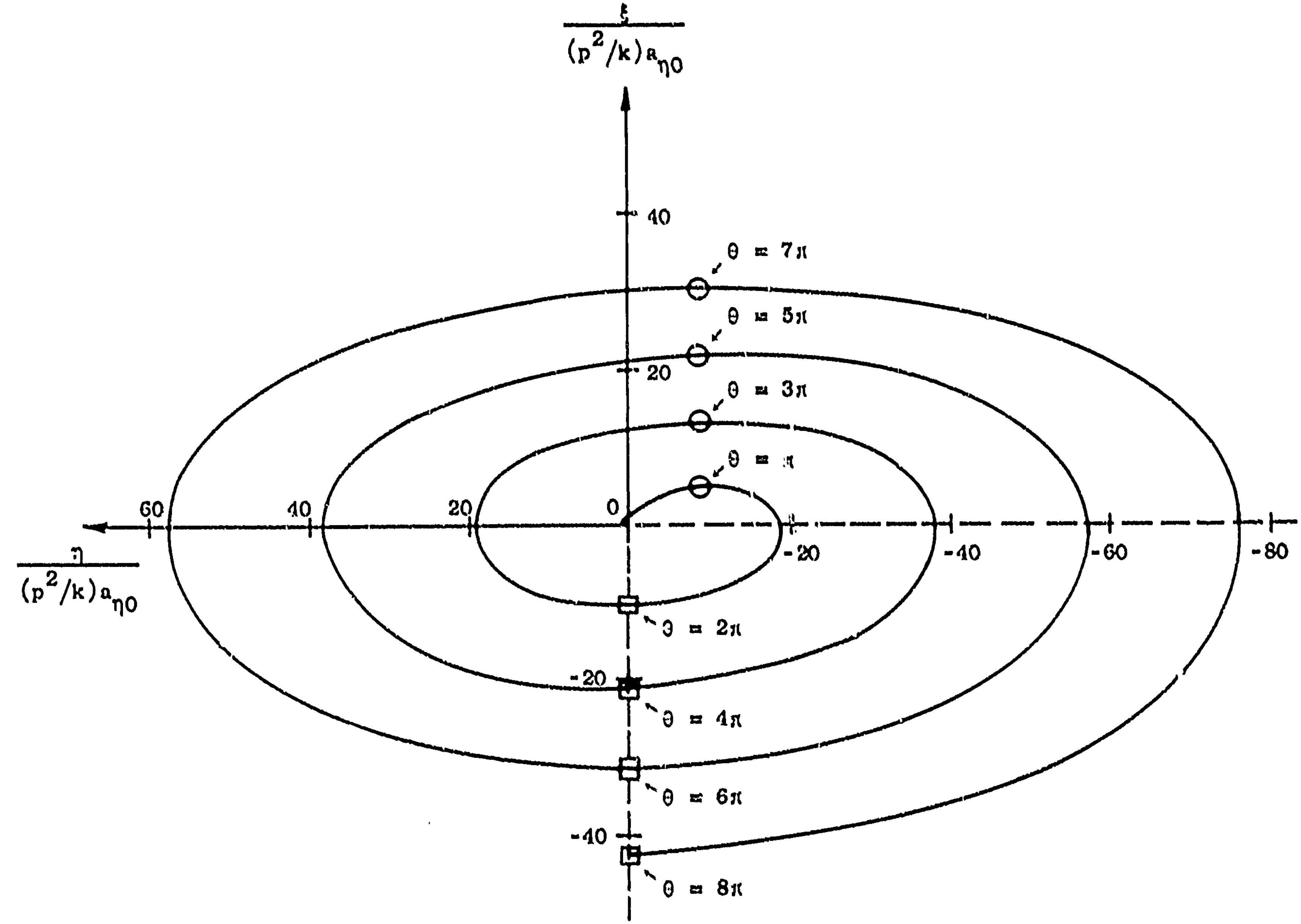


FIG. 10. ACCELERATION CONSTANT IN INERTIAL SPACE

$a_{\xi o} = 0$; $a_{\eta o} \neq 0$; $c = 0.1$

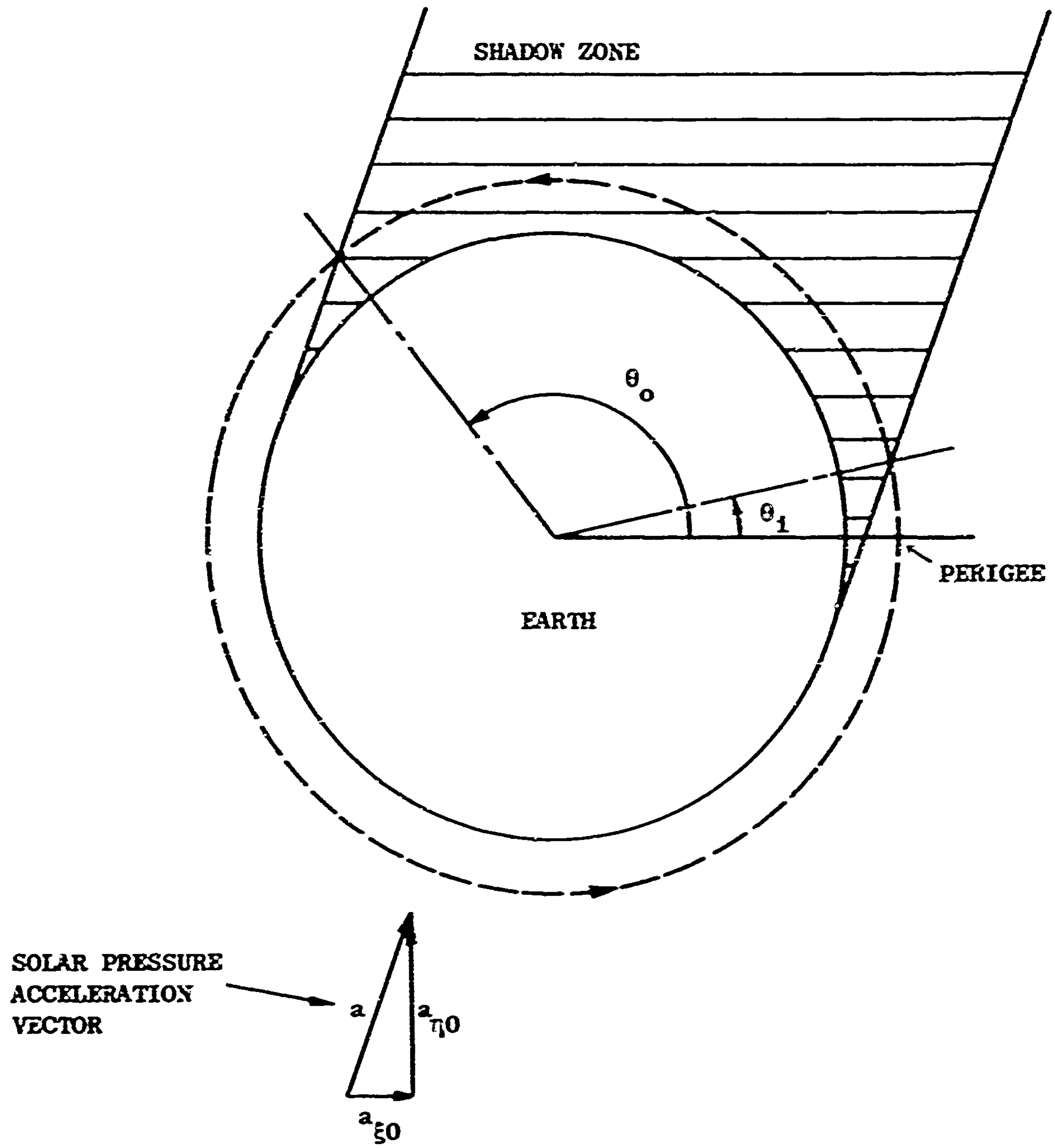


FIG. 11. SOLAR RADIATION WITH SHADOWING.

To determine

$$u(\theta) = \begin{pmatrix} P_1(\theta) \\ \\ P_2(\theta) \end{pmatrix}, \quad 0 \le \theta \le 2\pi \tag{86}$$

it is possible to expand $u(\theta)$ as a Fourier series. If this is done then

$$\begin{aligned}
P_1(\theta) &= \frac{1}{2\pi}\left\{a_{\xi o}(\sin\theta_i - \sin\theta_o) - a_{\eta o}(\cos\theta_i - \cos\theta_o)\right\} \\
&+ \frac{1}{2\pi}\left\{a_{\xi o}(2\pi + \theta_i - \theta_o) + a_{\xi o}\frac{\sin 2\theta_i - \sin 2\theta_o}{2} - a_{\eta o}\frac{\cos 2\theta_i - \cos 2\theta_o}{2}\right\}\cos\theta \\
&+ \frac{1}{2\pi}\left\{a_{\eta o}(2\pi + \theta_i - \theta_o) - a_{\eta o}\frac{\sin 2\theta_i - \sin 2\theta_o}{2} - a_{\xi o}\frac{\cos 2\theta_i - \cos 2\theta_o}{2}\right\}\sin\theta \\
&+ \frac{1}{2\pi}\sum_{n=2}^{\infty}\left\{a_{\xi o}\left(\frac{\sin(n+1)\theta_i - \sin(n+1)\theta_o}{(n+1)} + \frac{\sin(n-1)\theta_i - \sin(n-1)\theta_o}{(n-1)}\right)\right. \\
&\left. - a_{\eta o}\left(\frac{\cos(n+1)\theta_i - \cos(n+1)\theta_o}{(n+1)} - \frac{\cos(n-1)\theta_i - \cos(n-1)\theta_o}{(n-1)}\right)\right\}\cos n\theta \\
&- \frac{1}{2\pi}\sum_{n=2}^{\infty}\left\{a_{\xi o}\left(\frac{\cos(n+1)\theta_i - \cos(n+1)\theta_o}{(n+1)} + \frac{\cos(n-1)\theta_i - \cos(n-1)\theta_o}{(n-1)}\right)\right. \\
&\left. + a_{\eta o}\left(\frac{\sin(n+1)\theta_i - \sin(n+1)\theta_o}{(n+1)} - \frac{\sin(n-1)\theta_i - \sin(n-1)\theta_o}{(n-1)}\right)\right\}\sin n\theta \,.
\end{aligned} \tag{87}$$

$$
\begin{aligned}
P_2(\theta) = {} & \frac{1}{2\pi}\left\{a_{\eta o}(\sin\theta_i - \sin\theta_o) + a_{\xi o}(\cos\theta_i - \cos\theta_o)\right\} \\
& + \frac{1}{2\pi}\left\{a_{\eta o}(2\pi + \theta_i - \theta_o) + a_{\eta o}\frac{\sin 2\theta_i - \sin 2\theta_o}{2} + a_{\xi o}\frac{\cos 2\theta_i - \cos 2\theta_o}{2}\right\}\cos\theta \\
& + \frac{1}{2\pi}\left\{-a_{\xi o}(2\pi + \theta_i - \theta_o) + a_{\xi o}\frac{\sin 2\theta_i - \sin 2\theta_o}{2} - a_{\eta o}\frac{\cos 2\theta_i - \cos 2\theta_o}{2}\right\}\sin\theta \\
& + \frac{1}{2\pi}\sum_{n=2}^{\infty}\Bigg\{a_{\eta o}\left(\frac{\sin(n+1)\theta_i - \sin(n+1)\theta_o}{(n+1)} + \frac{\sin(n-1)\theta_i - \sin(n-1)\theta_o}{(n-1)}\right) \\
& + a_{\xi o}\left(\frac{\cos(n+1)\theta_i - \cos(n+1)\theta_o}{(n+1)} - \frac{\cos(n-1)\theta_i - \cos(n-1)\theta_o}{(n-1)}\right)\Bigg\}\cos n\theta \\
& - \frac{1}{2\pi}\sum_{n=2}^{\infty}\Bigg\{a_{\eta o}\left(\frac{\cos(n+1)\theta_i - \cos(n+1)\theta_o}{(n+1)} + \frac{\cos(n-1)\theta_i - \cos(n-1)\theta_o}{(n-1)}\right) \\
& - a_{\xi o}\left(\frac{\sin(n+1)\theta_i - \sin(n+1)\theta_o}{(n+1)} - \frac{\sin(n-1)\theta_i - \sin(n-1)\theta_o}{(n-1)}\right)\Bigg\}\sin n\theta
\end{aligned}
\tag{88}
$$

Figure (12) is a plot of perturbed motion over 4 orbit periods under the following conditions:

$\vec{a} = -a_{\xi o}$ (the sun lies along the line of apsides)

$e = 0.01$

$\theta_i = 135^\circ$

$\theta_o = 225^\circ$

In the numeric integration the Fourier expansion was carried out to the 19th term $(n = 19)$. It should be noted that the parameters selected

were chosen merely to provide an idea of the nature of the solution, rather than to describe some actual orbit condition. The problem of calculating actual shadow-entry and exit angles is discussed in the literature (cf. Reference 16) and is not within the scope of this paper.

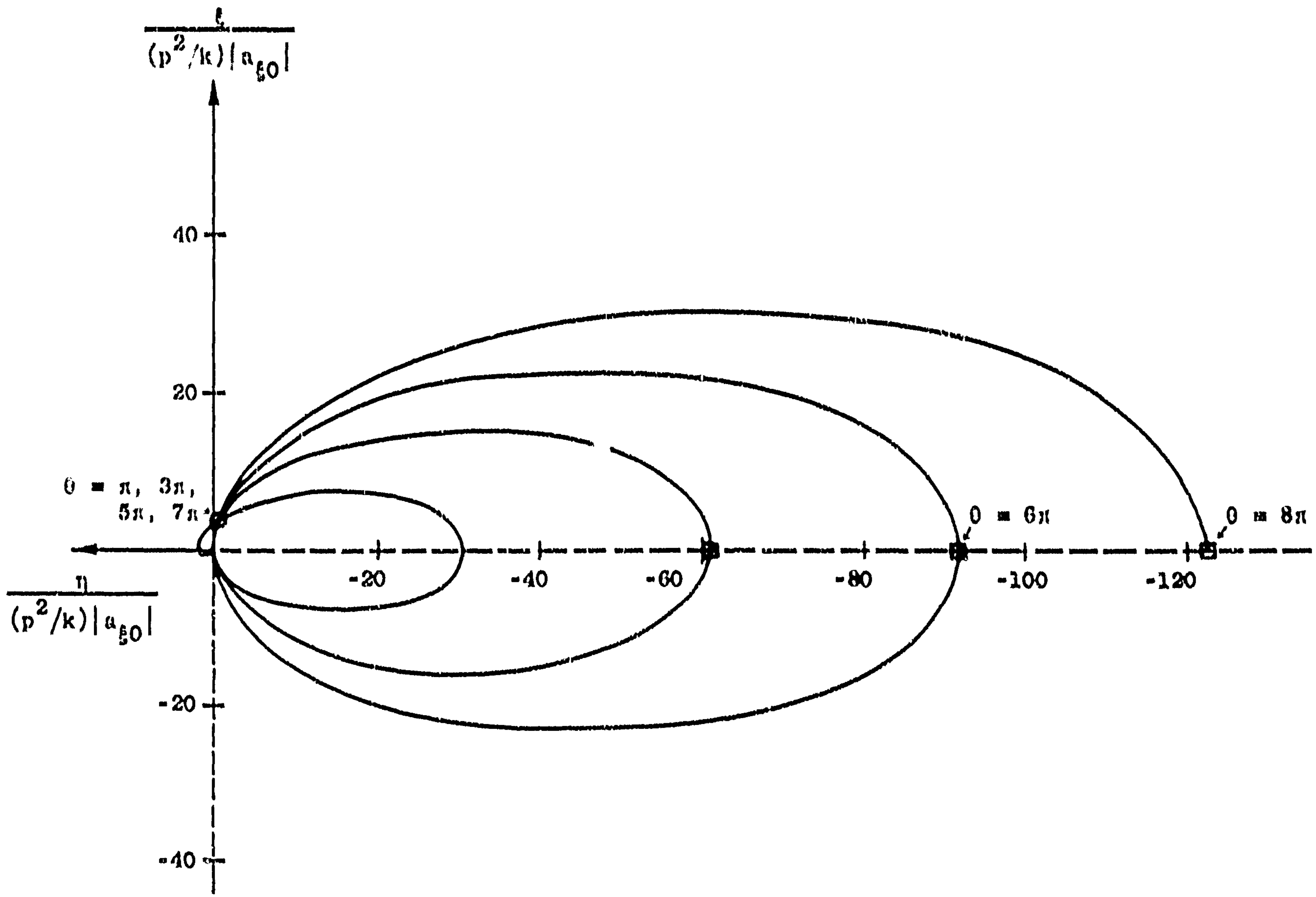


FIG. 12. EXAMPLE OF SOLAR RADIATION PERTURBATION EFFECT.

Application to Inertial Guidance

The basic relationship of inertial guidance is that geometric acceleration is equal to the output from an ideal accelerometer plus gravitational mass attraction. That is

$$\overset{II}{\vec{r}} = \vec{f} + \vec{g} \tag{89}$$

where $\vec{r}$ is the position vector of the vehicle,

$\vec{f}$ is the output of an ideal accelerometer on board the vehicle,

$\vec{g}$ is the gravitational mass attraction vector, and

overscript $(\overset{I}{\ })$ signifies d/dt in an inertial frame.

An inertial guidance system computer is mechanized such that it obtains the solution to equation (89) by solving the Ideal Mechanization Equations, (90) and (91).

$$\overset{c}{\vec{v}} = \vec{f} + \vec{g} - \vec{\omega} \times \vec{v} \tag{90}$$

$$\overset{c}{\vec{r}} = \vec{v} - \vec{\omega} \times \vec{r} \tag{91}$$

where $\vec{v} \triangleq \overset{I}{\vec{r}}$,

$\vec{\omega}$ is the angular velocity of the computer frame with respect to inertial space, and

overscript $(\overset{c}{\ })$ signifies d/dt in the computer frame.

It has been shown elsewhere[20] that from these three basic equations, by perturbation analysis, one obtains the Platform Misalignment Error Equation (92) and the Position and Velocity Error Equation (93) for an Inertial Navigation System in elliptical orbit

$$\overset{I}{\dot{\vec{\psi}}} = - \vec{\vec{K}}_g \cdot \vec{\omega} - \vec{\epsilon} \tag{92}$$

$$\delta\overset{II}{\ddot{\vec{r}}} + \omega_s^2 \delta\vec{r} - 3\omega_s^2 \frac{\vec{r} \cdot \delta\vec{r}}{r^2}\vec{r} = - \vec{\psi} \times \vec{f} + \vec{\vec{K}}_g \cdot \vec{f} + \vec{b} + \vec{\eta} - 2\delta\vec{\omega} \times \vec{v}$$
$$+ \vec{\vec{K}}_v \cdot \overset{c}{\vec{v}} + \frac{d^I}{dt}\left(\vec{\vec{K}}_p \cdot \overset{c}{\vec{r}}\right) - \delta\overset{I}{\dot{\vec{\omega}}} \times \vec{r} \tag{93}$$

where $\vec{\psi}$ is the vector approximating the small angle which rotates computer into platform axes,

$\vec{\vec{K}}_g$ is the diad representing stabilization gyro scale factor error,

$\vec{\epsilon}$ is the stabilization gyro drift rate or bias error vector,

$\delta\vec{r}$ is the first order approximation to the error in $\vec{r}$,

$\omega_s=(k/r^3)^{\frac{1}{2}}$ is the Shuler frequency corresponding to $|\vec{r}|$,

k is the universal gravitational field constant,

$\vec{b}$ represents accelerometer bias,

$\vec{\eta}$ represents random accelerometer errors,

$\delta\vec{\omega}$ is the difference between computer angular rate and platform angular rate (to first order),

$\vec{\vec{K}}_v$ is the diad representing first integrator scale factor error,

$\vec{\vec{K}}_p$ is the diad representing second integrator scale factor error, and

$\frac{d^I}{dt}$ signifies differentiation in an inertial frame.

Comparison of equation (93) with equation (A10) of Appendix A reveals the interesting fact that the homogeneous form of the Position and Velocity Error Equation of an Inertial Navigation System in elliptical orbit is identical to the homogeneous form of the Basic Perturbation Equation linearized about an elliptical orbit. It follows then that

equation (93) can be transformed to the Tschauner-Hempel Equations.

If

$$\delta\vec{r} \triangleq \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}, \tag{94}$$

coordinatized in a locally-level reference frame, and

$$\delta x \triangleq r\,\delta\xi; \quad \delta y \triangleq r\,\delta\eta; \quad \delta z \triangleq r\,\delta\zeta; \tag{95}$$

then equation (93) becomes equations (96) through (98):

$$\delta\xi'' - \frac{3}{1 + e\cos\theta}\,\delta\xi - 2\delta\eta' = \alpha \tag{96}$$

$$2\delta\xi' + \delta\eta'' = \beta \tag{97}$$

$$\delta\zeta'' + \delta\zeta = \gamma \tag{98}$$

where e is the eccentricity of the elliptical orbit in which the guidance system is operating,

θ is the true anomaly of the vehicle,

$\alpha = \dfrac{P_1}{\omega^2 r}$, $\beta = \dfrac{P_2}{\omega^2 r}$, $\gamma = \dfrac{P_3}{\omega^2 r}$

P_1, P_2, and P_3 are the coordinates of the error sources,

$\omega = \dot{\theta}$, the time rate of change of true anomaly, and

prime (') signifies $\dfrac{d}{d\theta} = \dfrac{1}{\omega}\dfrac{d}{dt}$.

If, for example, the accelerometers of the Inertial Navigation System are maintained in a local-level orientation, then accelerometer bias corresponds to a constant input to the equations of motion.

Figures (3) through (6) then may be interpreted as plots showing the propagation of system errors in nondimensional altitude and cross-track due to accelerometer bias, when

$$\vec{b} \triangleq \begin{bmatrix} a_\xi \\ a_\eta \end{bmatrix} . \tag{99}$$

The sampled-data solution discussed previously is of course valid too. Hence, if $a_\xi \approx 10^{-4} g_e$, $a_\eta \approx 10^{-4} g_e$, r = 100 miles plus Earth radius, e = 0.01, using equations (79) and (80) one obtains for N = 1 orbit,

$$\delta x = r\,\delta\xi \approx 4 \text{ miles} \tag{100}$$

$$\delta y = r\,\delta\eta \approx -10 \text{ miles} \tag{101}$$

APPENDIX A. DERIVATION OF THE TSCHAUNER-HEMPEL EQUATIONS

In this appendix the standard derivation of elliptical relative motion is reviewed for completeness and to establish notation. A derivation of the Tschauner-Hempel equations is also given.

Consider the relative motion between a reference object in an elliptical orbit, described by position vector $\vec{R}$, and a nearby object in a slightly different orbit, described by position vector $\vec{r}$. (See Figure 13). The relative position of the second object with respect to the first is designated by the vector $\vec{\rho}$ so that

$$\vec{R} + \vec{\rho} = \vec{r} \tag{A1}$$

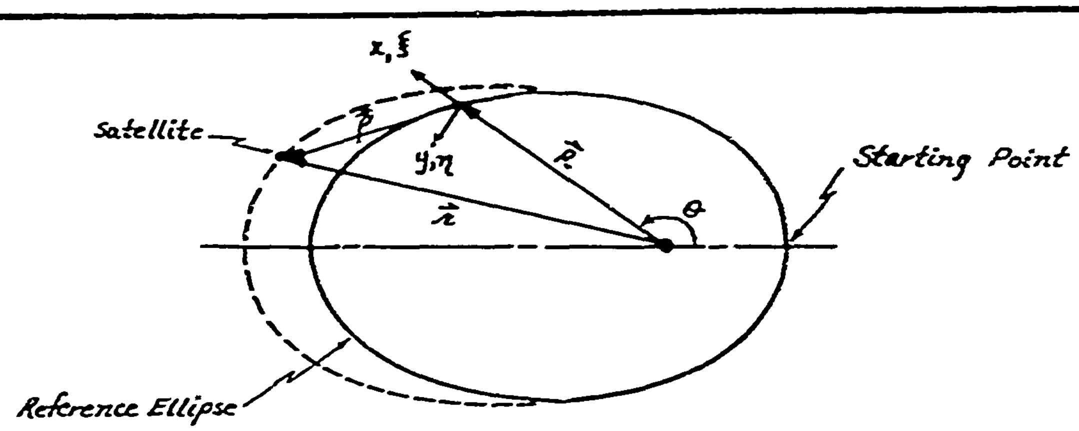


Figure 13. Coordinate System for Perturbation Equation

For simplicity assume both objects start together in space and time as shown. Considering that which makes the two orbits different to be a perturbing acceleration $\vec{a}$, the equations of motion can be written:

$$\overset{\text{II}}{\vec{R}} = - \frac{k\vec{R}}{|\vec{R}\,|^3} \quad \text{(unperturbed body)} \tag{A2}$$

$$\overset{II}{\vec{r}} = -\frac{k\vec{r}}{|\vec{r}|^3} + \vec{a} \quad \text{(perturbed body)} \tag{A3}$$

where k is the gravitational field constant, and superscript $(^{I})$ signifies differentiation with respect to time in an inertial frame. Equations (A1) and (A3) combine to form

$$\overset{II}{\vec{R}} + \overset{II}{\vec{\rho}} = \frac{-k(\vec{R} + \vec{\rho})}{|\vec{R} + \vec{\rho}|^3} + \vec{a} \tag{A4}$$

By taking the square root of the dot product of $(\vec{R} + \vec{\rho})$ with itself it is readily verified that

$$|\vec{R} + \vec{\rho}|^{-3} = [R^2(1 + \frac{2\vec{R} \cdot \vec{\rho}}{R^2} + \frac{\rho^2}{R^2})]^{-3/2} \tag{A5}$$

If terms of order $\left(\frac{\rho}{R}\right)^2$ are neglected as small compared with terms of order $\left(\frac{\rho}{R}\right)$,

$$|\vec{R} + \vec{\rho}|^{-3} \cong R^{-3} \left(1 + \frac{2\vec{R} \cdot \vec{\rho}}{R^2}\right)^{-3/2}$$

$$\cong R^{-3} \left[1 - \frac{3}{2} \left(\frac{2\vec{R} \cdot \vec{\rho}}{R^2}\right) + \text{higher order terms}\right] \tag{A6}$$

$$\therefore \quad |\vec{R} + \vec{\rho}|^{-3} \cong R^{-3} \left(1 - \frac{3\vec{R} \cdot \vec{\rho}}{R^2}\right) \tag{A7}$$

With equation (A7), equation (A4) becomes

$$\overset{II}{\vec{R}} + \overset{II}{\vec{\rho}} = -k(\vec{R} + \vec{\rho})R^{-3} \left(1 - \frac{3\vec{R} \cdot \vec{\rho}}{R^2}\right) + \vec{a} \tag{A8}$$

Subtract (A2) from (A8) to obtain

$$\overset{II}{\vec{\rho}} = \frac{3k}{R^3} \frac{\vec{R} \cdot \vec{\rho}}{R^2} \vec{R} - \frac{k}{R^3} \vec{\rho} + \frac{3k}{R^3} \frac{\vec{\rho} \cdot \vec{\rho}}{R^2} \vec{\rho} + \vec{a} \tag{A9}$$

Neglect the third term on the right-hand side as small compared with the first two, and the basic perturbation equation results

$$\overset{II}{\vec{\rho}} = - \frac{k}{R^3} \vec{\rho} + \frac{3k}{R^3} (\vec{R} \cdot \vec{\rho}) \vec{R} + \vec{a} \tag{A10}$$

If $\overset{L}{\vec{\rho}}$ is the time derivative of $\vec{\rho}$ taken in the rotating reference frame and $\vec{\omega}_{L/I}$ is the angular velocity vector of the rotating reference frame with respect to inertial space, then

$$\overset{I}{\vec{\rho}} = \overset{L}{\vec{\rho}} + \vec{\omega}_{L/I} \times \vec{\rho} \tag{A11}$$

and

$$\overset{II}{\vec{\rho}} = \overset{LL}{\vec{\rho}} + \overset{L}{\vec{\omega}}_{L/I} \times \vec{\rho} + 2\vec{\omega}_{L/I} \times \overset{L}{\vec{\rho}} + \vec{\omega}_{L/I} \times (\vec{\omega}_{L/I} \times \vec{\rho}) \tag{A12}$$

In the rotating reference frame, if we define

$$\xi \stackrel{\Delta}{=} \frac{x}{R} ; \quad \eta \stackrel{\Delta}{=} \frac{y}{R} ; \quad \zeta \stackrel{\Delta}{=} \frac{z}{R} , \tag{A13}$$

$$\vec{\rho} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} . \tag{A14}$$

Also in the rotating frame we have

$$\vec{R} = \begin{pmatrix} R \\ 0 \\ 0 \end{pmatrix} ; \quad \vec{a} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} ; \quad \vec{\omega}_{L/I} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} . \tag{A15}$$

Combining equations (A9) and (A12), resolved in the rotating frame, one obtains the scalar equations

$$\ddot{x} - \left(\omega^2 + \frac{2k}{R^3}\right)x - 2\omega\dot{y} - \dot{\omega}y = P_1 \tag{A16}$$

$$\ddot{y} - \left(\omega^2 - \frac{k}{R^3}\right)y + 2\omega\dot{x} + \dot{\omega}x = P_2 \tag{A17}$$

$$\ddot{z} + \frac{k}{R^3} z = P_3 \tag{A18}$$

where $(\dot{\ })$ signifies differentiation with respect to time.

The following identities can be obtained by differentiation:

$$\omega R\xi' = \dot{x} - \frac{e\omega \sin\theta}{1 + e\cos\theta} x \tag{A19}$$

$$\omega^2 R\xi'' = \ddot{x} + \left(\frac{k}{R^3} - \omega^2\right)x \tag{A20}$$

where $(\ ')$ signifies $\frac{d}{d\theta} = \frac{1}{\omega}\frac{d}{dt}$.

Expressions identical in form hold for η and ζ. Combining these equations with equations (A16) through (A18), and noting that

$$\dot{\omega} = \omega\,\omega' = -\frac{2\omega^2 e \sin\theta}{1 + e\cos\theta} \tag{A21}$$

yields

$$\omega^2 R\xi'' - \frac{3k}{R^2}\xi - 2\omega^2 R\eta' = P_1 \tag{A22}$$

$$\omega^2 R\eta'' + 2\omega^2 R\xi' = P_2 \tag{A23}$$

$$\omega^2 R\zeta'' + w^2 R\zeta = P_3 \tag{A24}$$

Noting that

$$\frac{k}{R^2} = \frac{\omega^2 R}{1 + e \cos \theta} \tag{A25}$$

the Tschauner-Hempel equations are obtained:

$$\xi'' - \frac{3\xi}{1 + e \cos \theta} - 2\eta' = \frac{1}{\omega^2 R} P_1 \tag{A26}$$

$$\eta'' + 2\xi' = \frac{1}{\omega^2 R} P_2 \tag{A27}$$

$$\zeta'' + \zeta = \frac{1}{\omega^2 R} P_3 \tag{A28}$$

APPENDIX B. REVIEW OF FLOQUET THEORY

In this appendix the standard results of the theory of linear differential equations[17,18,19] are reviewed for completeness and to establish the notation.

Theorem: The n^{th}-order linear inhomogeneous system

$$x'(\theta) = F(\theta)x(\theta) + D(\theta)u(\theta) \; ; \quad x(\theta_o) = x_o \qquad \text{(B1)}$$

has the general solution

$$x(\theta) = X(\theta, \theta_o)x_o + X(\theta, \theta_o) \int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \qquad \text{(B2)}$$

where $X(\theta, \theta_o)$, the $n \times n$ state transition matrix, is the solution of

$$X'(\theta, \theta_o) = F(\theta)X(\theta, \theta_o) \; ; \quad X(\theta_o, \theta_o) = U \quad \text{(the unit matrix)} \qquad \text{(B3)}$$

Proof 1: Substitute B2 into B1.

Proof 2: Assume the solution $x(\theta)$ to be made up of the complementary solution $x_c(\theta)$ and a particular solution $x_p(\theta)$:

$$x(\theta) = x_c(\theta) + x_p(\theta) \qquad \text{(B4)}$$

The n^{th}-ordered homogeneous form of equation (B1) has n special linearly independent solutions which can be arranged as columns of an $n \times n$ matrix $X(\theta, \theta_o)$* which satisfies equation (B3). $X(\theta_o, \theta_o)$ was chosen as the

*$X(\theta, \theta_o)$ is known as the matrizant, fundamental matrix, state transition matrix, or matrix of partials.

unit matrix so that an arbitrary complementary solution would [illegible]

$$x_c(\theta) = X(\theta, \theta_o)x_o \tag{B5}$$

In order to obtain the particlar solution assume the constants, x_o, of the homogeneous solution are now functions of θ, and call these functions $c(\theta)$.

$$x_p(\theta) = X(\theta, \theta_o)c(\theta) \tag{B6}$$

This apparently arbitrary assumption was first made by Lagrange and was motivated by a desire to represent the effects of planetary perturbations in the solar system as variation of the orbit elements. This assumption, it turns out, gives the exact solution for the special case of linear equations. When (B6) is substituted into (B1) we obtain

$$X'c + Xc' = FXc + Du \tag{B7}$$

or

$$c' = X^{-1}\,Du \tag{B8}$$

since $X^{-1} = FX$. Equation (B8) may be integrated immediately to obtain

$$c(\theta) = \int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \tag{B9}$$

proving equation (B2).

If F in equation (B1) is a constant matrix then it can be seen that

$$X(\theta, \theta_o) = \epsilon^{F(\theta-\theta_o)} \tag{B10}$$

where for an arbitrary $n \times n$ matrix A,

$$\epsilon^{A} \triangleq \sum_{R=0}^{\infty} \frac{1}{R!} A^{R} \tag{B11}$$

Lemma: If in the system (B1), $F(\theta) = F(\theta + 2\pi)$, then for any integer R,

$$X(\theta + 2\pi R, \theta_o) = X(\theta, \theta_o) X^{R}(\theta_o + 2\pi, \theta_o) \tag{B12}$$

Proof: $X'(\theta, \theta_o) \triangleq F(\theta)X(\theta, \theta_o)$; $X(\theta_o, \theta_o) = U$ (the unit matrix) (B3). Since this must hold for all θ,

$$\begin{aligned} X'(\theta + 2\pi, \theta_o) &= F(\theta + 2\pi)X(\theta + 2\pi, \theta_o) \\ &= F(\theta)X(\theta + 2\pi, \theta_o) \end{aligned} \tag{B13}$$

since $F(\theta) = F(\theta + 2\pi)$. The columns of $X(\theta + 2\pi, \theta_o)$ are n linearly independent solutions of the homogeneous part of (B1), and therefore, each of these columns, $x_R (1 \leq R \leq n)$, is given by $x_R = X(\theta, \theta_o)c_R$ where for each R, c_R is an $n \times 1$ column matrix of constants. Let C be an $n \times n$ matrix whose columns are the c_R. Then

$$X(\theta + 2\pi, \theta_o) = X(\theta, \theta_o)C \tag{B14}$$

Since the columns of $X(\theta + 2\pi, \theta_o)$ are independent, C^{-1} exists. Equation (B14) must hold for all θ. Specifically it must hold for $\theta = \theta_o$:

$$X(\theta_o + 2\pi, \theta_o) = X(\theta_o, \theta_o)C \tag{B15}$$

Since $X(\theta_o, \theta_o) \triangleq U$, C is known and

$$X(\theta + 2\pi, \theta_o) = X(\theta, \theta_o)X(\theta_o + 2\pi, \theta_o) \tag{B16}$$

Equation (B16) must also hold for all θ. Specifically it must hold for $\theta = \theta + 2\pi$:

$$X(\theta + 4\pi, \theta_o) = X(\theta + 2\pi, \theta_o)X(\theta_o + 2\pi, \theta_o)$$

$$= X(\theta, \theta_o)X^2(\theta_o + 2\pi, \theta_o) \qquad (B17)$$

By induction,

$$X(\theta + 2\pi R, \theta_o) = X(\theta, \theta_o)X^R(\theta_o + 2\pi, \theta_o) \qquad (B12)$$

For the balance of the discussion it will be assumed that $F(\theta) = F(\theta + 2\pi)$.

Define a matrix $R(\theta, \theta_o)$ by

$$R(\theta, \theta_o) \triangleq X(\theta, \theta_o)\epsilon^{-B(\theta-\theta_o)} \qquad (B18)$$

where B is a constant $n \times n$ matrix not yet specified. Then

$$X(\theta, \theta_o) = R(\theta, \theta_o)\epsilon^{B(\theta-\theta_o)} \qquad (B19)$$

(note the similarity with equation (B10)). Then using (B12):

$$R(\theta + 2\pi, \theta_o)\epsilon^{B(\theta+2\pi-\theta_o)} = R(\theta, \theta_o)\epsilon^{B(\theta-\theta_o)} R(\theta_o + 2\pi, \theta_o)\epsilon^{B2\pi} \qquad (B20)$$

Now define B to be

$$B \triangleq \frac{1}{2\pi} \ln X(\theta_o + 2\pi, \theta_o) \qquad (B21)$$

then from (B19) and (B21):

$$X(\theta_o + 2\pi, \theta_o) = \epsilon^{B2\pi} = R(\theta_o + 2\pi, \theta_o)\epsilon^{B2\pi} \qquad (B22)$$

$$\therefore \ R(\theta_o + 2\pi, \theta_o) = U \quad \text{(the unit matrix)} \qquad (B23)$$

When (B23) is substituted into (B20) we obtain $R(\theta + 2\pi, \theta_o) = R(\theta, \theta_o)$, so that $R(\theta, \theta_o)$ is a periodic matrix. Now let

$$W \triangleq \epsilon^{B(\theta-\theta_o)} \Rightarrow W' = BW \tag{B24}$$

Then $X' = FX$ implies

$$R'W + RW' = FRW \tag{B25}$$

$$R'W + RBW = FRW \tag{B26}$$

$$\therefore \quad R' = FR - RB \tag{B27}$$

and

$$B = R^{-1} FR - R^{-1}R' \tag{B28}$$

This result, where F and R are periodic and B is constant, is called the Lyapunov reduction of equation (B1).

Let Λ be the Jordan canonical form of B; i.e.

$$\Lambda = QBQ^{-1} \tag{B29}$$

then

$$W = \epsilon^{B(\theta-\theta_o)} = \epsilon^{Q^{-1}\Lambda Q(\theta-\theta_o)} = Q^{-1}\epsilon^{\Lambda(\theta-\theta_o)}Q \tag{B30}$$

and

$$X(\theta, \theta_o) = R(\theta, \theta_o)Q^{-1}\epsilon^{\Lambda(\theta-\theta_o)}Q \tag{B31}$$

If the transformation $P(\theta)$ is introduced so that

$$z(\theta) = P^{-1}(\theta)x(\theta) \tag{B32}$$

where

$$z'(\theta) = \Lambda z(\theta) \; ; \quad z(\theta_o) = P^{-1}(\theta_o)x_o \tag{B33}$$

then

$$z(\theta) = \epsilon^{\Lambda(\theta-\theta_o)} z(\theta_o) = \epsilon^{\Lambda(\theta-\theta_o)} P^{-1}(\theta_o)x_o \tag{B34}$$

Combining (B32) and (B34):

$$x(\theta) = P(\theta)\epsilon^{\Lambda(\theta-\theta_o)} P^{-1}(\theta_o)x_o \tag{B35}$$

From (B35) it follows that

$$X(\theta, \theta_o) = P(\theta)\epsilon^{\Lambda(\theta-\theta_o)} P^{-1}(\theta_o) \tag{B36}$$

If we take

$$Q \triangleq P^{-1}(\theta_o) \tag{B37}$$

then it follows from (B31) that

$$\begin{aligned} R(\theta, \theta_o) &= P(\theta)Q \\ &= P(\theta)P^{-1}(\theta_o) \end{aligned} \tag{B38}$$

Equation (B36) is the form of the state transition matrix used in the basic text.

APPENDIX C. DERIVATION OF A SPECIAL FORM OF THE SOLUTION TO LINEAR EQUATIONS WITH PERIODIC COEFFICIENTS

The solution to

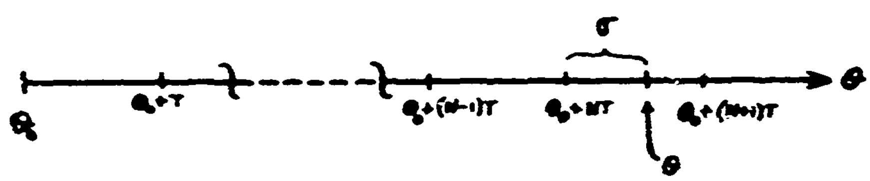
$$x'(\theta) = F(\theta)x(\theta) + D(\theta)u(\theta) \; ; \quad x(\theta_o) = x_o \qquad \text{(C1)}$$

has been shown to be (see Appendix B)

$$x(\theta) = X(\theta, \theta_o)x_o + X(\theta, \theta_o) \int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \qquad \text{(C2)}$$

[illegible] such that

$$\sigma \stackrel{\Delta}{=} \theta - NT - \theta_o \qquad \text{(C3)}$$



Lemma: If $F(\theta) = F(\theta + T)$ then

$$X(\theta, \theta_o) = X(\theta_o + \sigma, \theta_o)X^N(\theta_o + T, \theta_o) \qquad \text{(C4)}$$

Proof: In Appendix B it was established (for $T = 2\pi$, no restriction) that

$$X(\theta + NT, \theta_o) = X(\theta, \theta_o)X^N(\theta_o + T, \theta_o) \qquad \text{(C5)}$$

Let

$$\theta = \theta + NT \qquad \text{(C6)}$$

then

$$X(\theta, \theta_o) = X(\theta - NT, \theta_o)X^N(\theta_o + T, \theta_o) \tag{C7}$$

From (C3)

$$X(\theta, \theta_o) = X(\theta_o + \sigma, \theta_o)X^N(\theta_o + T, \theta_o) \tag{C8}$$

Substituting relation (C8) into equation (C2) yields.

$$x(\theta) = X(\theta_o + \sigma, \theta_o)X^N(\theta_o + T, \theta_o)x_o$$

$$+ X(\theta_o + \sigma, \theta_o)X^N(\theta_o + T, \theta_o) \int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \tag{C9}$$

Lemma: If $D(\theta) = D(\theta + T)$ and $u(\theta) = u(\theta + T)$ the solution (C9) can be written

$$x(\theta) = X(\theta_o + \sigma, \theta_o)C^N x_o + X(\theta_o + \sigma, \theta_o)\left(\sum_{k=1}^{N} C^k\right) I_1$$

$$+ X(\theta_o + \sigma, \theta_o) \int_{\theta_o}^{\theta_o+\sigma} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \tag{C10}$$

where
$$C = X(\theta_o + T, \theta_o) \tag{C11}$$

$$I_1 = \int_{\theta_o}^{\theta_o+T} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \tag{C12}$$

Proof: $$\int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau = \int_{\theta_o}^{\theta_o+T} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau$$

$$+ \int_{\theta_o+T}^{\theta_o+2T} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau +$$

$$+ \ldots + \int_{\theta_o+(N-1)T}^{\theta_o+NT} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau$$

$$+ \int_{\theta_o+NT}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \ .$$

By simple changes of variable in each integral obtain

$$\int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau = \int_{\theta_o}^{\theta_o+T} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau +$$

(see page 59)

Proof: $$\int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau = \int_{\theta_o}^{\theta_o+T} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau$$

$$+ \int_{\theta_o+T}^{\theta_o+2T} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau +$$

$$+ \dots + \int_{\theta_o+(N-1)T}^{\theta_o+NT} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau$$

$$+ \int_{\theta_o+NT}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \; .$$

By simple changes of variable in each integral obtain

$$\int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau = \int_{\theta_o}^{\theta_o+T} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau +$$

(see page 59)

$$+ \int_{\theta_o}^{\theta_o + T} X^{-1}(\tau + T, \theta_o) D(\tau) u(\tau) d\tau +$$

$$+ \ldots\ldots + \int_{\theta_o}^{\theta_o + T} X^{-1}(\tau + (N - 1)T, \theta_o) D(\tau) u(\tau) d\tau$$

$$+ \int_{\theta_o}^{\theta - NT} X^{-1}(\tau + NT, \theta_o) D(\tau) u(\tau) d\tau$$

Use $X(\tau + NT, \theta_o) = X(\theta, \theta_o) X^{N}(\theta_o + T, \theta_o)$ shown above to obtain

$$X^{-1}(\tau + NT, \theta_o) = X^{-N}(\theta_o + T, \theta_o) X^{-1}(\tau, \theta_o)$$

$$\int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o) D(\tau) u(\tau) d\tau = \int_{\theta_o}^{\theta_o + T} X^{-1}(\tau, \theta_o) D(\tau) u(\tau) d\tau$$

$$+ X^{-1}(\theta_o + T, \theta_o) \int_{\theta_o}^{\theta_o + T} X^{-1}(\tau, \theta_o) D(\tau) u(\tau) d\tau +$$

$$+ \ldots\ldots + X^{-(N-1)}(\theta_o + T, \theta_o) \int_{\theta_o}^{\theta_o + T} X^{-1}(\tau, \theta_o) D(\tau) u(\tau) d\tau$$

$$+ X^{-N}(\theta_o + T, \theta_o) \int_{\theta_o}^{\theta_o + \sigma} X^{-1}(\tau, \theta_o) D(\tau) u(\tau) d\tau$$

Introduce this last relationship into equation (C9) and simplify to obtain expressions (C10) through (C12). Thus are proven expressions (36) and (37) of the basic text.

Lemma: If $D(\theta) = D(\theta + T)$ and $u(\theta) = u(\theta + MT)$ where M is an integer the solution (equation (C9)) can be written

$$x(\theta) = X(\theta_o + \sigma, \theta_o)C^N x_o + X(\theta_o + \sigma, \theta_o)\left(\sum_{k=0}^{r-1} C^{N-kM}\right) I_2$$

$$+ X(\theta_o + \sigma, \theta_o)C^{N-rM} \int_{\theta_o}^{\theta - rMT} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \qquad \text{(C13)}$$

where $C = X(\theta_o + T, \theta_o)$ (C14)

$$I_2 = \int_{\theta_o}^{\theta_o + MT} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau \qquad \text{(C15)}$$

r is an integer such that $rM \leq N \leq (r + 1)M$ (C16)

Proof: $$\int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau = \int_{\theta_o}^{\theta_o + MT} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau + \dots \dots$$

$$\dots + \int_{\theta_o + (r-1)MT}^{\theta_o + rMT} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau + \int_{\theta_o + rMT}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau .$$

where r is described by equation (C16) and N is still the largest number of integer values of T in θ. Again use simple variable changes in the integrals to obtain

$$\int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau = \int_{\theta_o}^{\theta_o+MT} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau + \ldots\ldots$$

$$\ldots\ldots + \int_{\theta_o}^{\theta_o+MT} X^{-1}(\tau + (r-1)MT, \theta_o)D(\tau)u(\tau)d\tau +$$

$$+ \int_{\theta_o}^{\theta-rMT} X^{-1}(\tau + rMT, \theta_o)D(\tau)u(\tau)d\tau$$

Again $X^{-1}(\tau + NT, \theta_o) = X^{-N}(\theta_o + T, \theta_o)X^{-1}(\tau, \theta_o)$, so that

$$\int_{\theta_o}^{\theta} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau = \int_{\theta_o}^{\theta_o+MT} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau +$$

$$+ X^{-M}(\theta_o + T, \theta_o) \int_{\theta_o}^{\theta_o+MT} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau +$$

$$+ \ldots\ldots X^{-(r-1)M}(\theta_o + T, \theta_o) \int_{\theta_o}^{\theta_o+MT} X^{-1}(\tau, \theta_o)D(\tau)u(\tau)d\tau +$$

(see page 62)

$$+ X^{-rM}(\theta_o + T, \theta_o) \int_{\theta_o}^{\theta - rMT} X^{-1}(\tau, \theta_o) D(\tau) u(\tau) d\tau \ .$$

Substitution into equation (C9) and simplification yields relations (C13) through (C15). Thus equations (39) and (40) of the basic text are proven.

Lemma: If $D(\theta) = D(\theta + T)$ and $u(\theta) = u(\theta + P)$ where $P \neq MT$ for $M = 0, 1, 2, \ldots\ldots$, then define K such that $KP \cong MT$, where K and M are both integers. Then $x(\theta)$ may be approximated by:

$$x(\theta) = X(\theta_o + \sigma, \theta_o) C^N x_o + X(\theta_o + \sigma, \theta_o) \left(\sum_{k=0}^{r-1} C^{N-kM} \right) I_3$$

$$+ X(\theta_o + \sigma, \theta_o) C^{N-rM} \int_{\theta_o}^{\theta - rKP} X^{-1}(\tau, \theta_o) D(\tau) u(\tau) d\tau \qquad \text{(C17)}$$

where $C = X(\theta_o + T, \theta_o)$

$$KP \cong MT, \ rKP \leq NT < (r-1)KP \qquad \text{(C19)}$$

$$I_3 = \int_{\theta_o}^{\theta_o + KP} X^{-1}(\tau, \theta_o) D(\tau) u(\tau) d\tau \qquad \text{(C20)}$$

Proof: The proof of relationships (C17) through (C20) [illegible] the proof for relationships (C13) through (C16) directly.

APPENDIX D. THE I-MATRIX

The sampled-data solution defined by equation (73), that is

$$x(2\pi N) = P(0)J^N P^{-1}(0)x_0 + P(0)SP^{-1}(0)I \tag{D1}$$

is (except for the I-matrix) composed of matrices whose closed forms are given in the main body of the text. The I-matrix, representing the integral (over 2π) of the disturbance, is in analytic form

$$I = \int_0^{2\pi} X^{-1}(\tau,0)D(\tau)u(\tau)d\tau. \tag{D2}$$

For the case of disturbances constant in the rotating reference frame, that is disturbances of the form

$$u(\theta) = \begin{pmatrix} a_\xi \\ a_\eta \end{pmatrix} \tag{D3}$$

the I-matrix has been computed in closed form. The technique employed in this calculation was primarily one of contour integration. The result is:

$$I = \frac{1}{k/p^2} \begin{pmatrix} \dfrac{\pi(4-e)}{(1-e)(1-e^2)^{3/2}} a_\eta \\[2ex] -\dfrac{3e\pi(e^2+2e+2)}{(1-e)(1-e^2)^{5/2}} a_\xi + \dfrac{6e\pi^2}{(1-e)(1-e^2)^2} a_\eta \\[2ex] -\dfrac{\pi(e^2+10e+4)}{(1-e)^2(1-e^2)^{3/2}} a_\xi + \dfrac{6\pi^2}{(1-e)^2(1-e^2)} a_\eta \\[2ex] \dfrac{3\pi(e^2-2e-2)}{(1-e^2)^{5/2}} a_\eta \end{pmatrix} \tag{D4}$$

where p is the semilatisrectum of the reference (nominal) ellipse,

k is the gravitational field constant, and

e is the eccentricity of the reference ellipse.

For disturbances of the form

$$u(\theta) = \begin{pmatrix} c_1 \epsilon^{jK_1\theta} \\ \\ c_2 \epsilon^{jK_2\theta} \end{pmatrix} \tag{D5}$$

where K_1 and K_2 are integers,

c_1 and c_2 are complex constants,

the I-matrix becomes

$$I = \frac{1}{k/p^2} \begin{pmatrix} (1+e)f_1 - \frac{(2+e)}{(1+e)} f_2 \\ \\ \frac{1}{(1-e)(1-e^2)^{3/2}} \left\{3ef_3 - f_4\right\} \\ \\ \frac{1}{e(1-e)^2 \quad 1-e^2} \left\{3ef_3 - \left[1 - (1-e)^2 \sqrt{1-e^2}\ \right] f_4\right\} \\ \\ -(2+e)\, f_1 + \frac{(3+e)}{(1+e)} f_2 \end{pmatrix} \tag{D6}$$

where p, k, and e are as defined in the previous case, and

$$f_1 = \frac{\pi}{16e(1-e^2)^{5/2}} \Big\{ j32c_1(1-e^2)^2 K_1 Z_1^{K_1} - e^3 c_2 Z_1^{K_2-4} \big[e(K_2^2+K_2)Z_1^8$$

$$+ 4(K_2^2-K_2)Z_1^7 - 2e(7K_2-2)Z_1^6 - 4(K_2^2+9K_2-12)Z_1^5 - 2e(K_2^2-20)Z_1^4 \tag{D7}$$

$$- 4(K_2^2-9K_2-12)Z_1^3 + 2e(7K_2+2)Z_1^2 + 4(K_2^2+K_2)Z_1 + e(K_2^2-K_2) \big] \Big\}$$

$$f_2 = \frac{2\pi}{\sqrt{1-e^2}} \left\{ jc_1K_1Z_1^{K_1} - c_2Z_1^{K_2} \right\} \tag{D8}$$

$$\begin{aligned} f_4 = -\frac{\pi}{4e(1-e^2)^{3/2}} \Big\{ & 2e^2c_1Z_1^{K_1-2}(K_1Z_1^4 - 4Z_1^2 - K_1) + jc_2K_2Z_1^{K_2-2}[e^2K_2Z_1^4 \\ & + 4e(K_2-1)Z_1^3 + 4(1-2e^2)Z_1^2 - 4e(K_2+1)Z_1 - e^2K_2] \Big\} \end{aligned} \tag{D9}$$

$$\begin{aligned} f_3 = -\frac{2\pi}{\sqrt{1-e^2}} \Big\{ & c_1Z_1^{K_1}(1+K_1\ln|Z_1|+jK_1\pi) + c_1K_1S_1 + c_2Z_1^{K_2}(j\ln|Z_1|-\pi) \\ & + jc_2S_2(K_2) \Big\} + \frac{2\pi}{1+e} \Big\{ c_1 + 2(1+e)[jc_1K_1S_2(K_1) - c_2S_2(K_2)] \Big\} \\ & - \frac{\pi}{48e^2(1-e^2)^{5/2}} \Big\{ e^3c_1Z_1^{K_1-4} \Big([(8e+4e^3)\sqrt{1-e^2}][(K_1^2-3K_1+2)Z_1^6 \\ & - (2K_1^2-8)Z_1^4 + (K_1^2+3K_1+2)Z_1^2] + [2 - 2(1-e^2)^{3/2}][(K_1^2-K_1)Z_1^7 \\ & - (K_1^2+9K_1-12)Z_1^5 - (K_1^2-9K_1-12)Z_1^3 + (K_1^2+K_1)Z_1] + \\ & + [e - (e-2e^3)\sqrt{1-e^2}]\,[(K_1^2+K_1)Z_1^8 - (14K_1-4)Z_1^6 - (2K_1^2-40)Z_1^4 \\ & + (14K_1+4)Z_1^2 + (K_1^2-K_1)] + [24e(1-e^2)]\,[(K_1-2)Z_1^6 \\ & - (2K_1-2)Z_1^4 - K_1Z_1^2] \Big) - j4c_2(1-e^2)K_2Z_1^{K_2-4} \Big([(4+2e^2)\sqrt{1-e^2} \\ & - 4]\,[e(K_2-1)Z_1^3 - e(K_2+1)Z_1] + [(1+2e^2)\sqrt{1-e^2} - 1]\,[e^2K_2Z_1^4 \\ & + (4-8e^2)Z_1^2 - e^2K_2] \Big) \Big\} \end{aligned} \tag{D10}$$

$$S_1 = \sum_{k=0}^{r-1} \frac{r!}{(r-k)!\,k!\,(r-k)} \left\{ Z_1^k(1-Z_1)^{r-k} - Z_1^k(-Z_1)^{r-k} - Z_2^k(1-Z_2)^{r-k} + Z_2^k(-Z_2)^{r-k} \right\} + Z_1^r \ln\left(\frac{Z_1-1}{Z_1}\right) - Z_2^r\left(\frac{Z_2-1}{Z_2}\right) \qquad \text{(D11)}$$

$$S_2(K_1) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} \left(\frac{je}{2}\right)^{2k+1} \times$$

$$\left\{ \left[\frac{1}{(2k+1)!} \frac{d^{2k+1}}{dZ^{2k+1}} \left(\frac{(Z^2-1)^{2k+1}(Z_1Z^2-2Z+Z_1)^{2k+1} Z^{K_1-2k+1}}{e^{2k+2}(Z-Z_2)^{2k+2}}\right)\right]_{Z=Z_1} \right. \qquad \text{(D12)}$$

$$\left. + \left[\frac{1}{(2k-K_1)!} \frac{d^{2k-K_1}}{dZ^{2k-K_1}} \left(\frac{(Z^2-1)^{2k+1}(Z_1Z^2-2Z+Z_1)^{2k+1}}{(eZ^2+2Z+e)^{2k+2}}\right)\right]_{\substack{Z=0 \\ K_1 \le 2k}} \right\} ,$$

$$Z_1 = -\frac{1}{e}\left(1 - \sqrt{1-e^2}\right) , \qquad \text{(D13)}$$

$$Z_2 = -\frac{1}{e}\left(1 + \sqrt{1-e^2}\right) . \qquad \text{(D14)}$$

The infinite series S_2 in the f_3 term above arises in the evaluation of

$$S_2(K) = \frac{1}{4\pi} \int_0^{2\pi} \frac{\sin^{-1}\lambda}{(1 + e\cos\tau)} \epsilon^{jK\tau} d\tau \qquad \text{(D15)}$$

where $\lambda = \dfrac{e \sin\tau\,(1 - Z_1 \cos\tau)}{1 + e\cos\tau}$. (D16)

An approximation to this integral can be made by observing that

$$\lambda = e \sin\tau \; [1 - (Z_1+e) \cos\tau + e(Z_1+e) \cos^2\tau - e^2(Z_1+e)\cos^3\tau + \ldots.] \quad \text{(D17)}$$

and

$$\sin^{-1}\lambda = \lambda + \frac{1}{6}\lambda^3 + \ldots. + \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} \lambda^{2k+1} + \ldots. \quad . \quad \text{(D18)}$$

Since Z_1 and e are of the same order of magnitude, λ may be approximated to whatever accuracy desired by cutting off the series and discarding similar powers of e in λ^3, λ^5, etc. Term by term integration can then be accomplished on the unit circle.

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